

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) your instructor's name, and (4) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (40 points)

a. Find a formula of the form

$$\int_0^{2\pi} f(x) dx = A_1 f(0) + A_2 f(\pi)$$

that is exact for any function having the form

$$f(x) = a + b \cos x ; \quad a, b \in \mathbf{R}$$

b. Prove that the resulting formula is exact for any function of the form

$$f(x) = \sum_{k=0}^n a_k \cos(2k+1)x$$

c. Assume that f is an even function, 2π -periodic, and satisfying

$$f(x) = \pi - x ; \quad \forall x \in [0, \pi]$$

i. Construct the Fourier series corresponding to the function f .

ii. Find the value of $S = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$

iii. Apply the formula found in question a. to calculate the integral of the obtained Fourier series from 0 to 2π . Is the integration exact?

2. (30 points) Count the number of multiplications needed for computing the discrete Fourier transform (DFT) when using the fast Fourier transform (FFT) in the case where $N = 2^p$ with $p \in \mathbf{N}$.

HEY, THERE'S MORE—TURN THE PAGE OVER!

3. (30 points)

- a. Let $f \in \mathcal{C}^3([0, 1])$. We want to use the central difference quotient with step size h to approximate $f'(x)$ i.e.

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

- i. Find the error $\mathcal{E}(x, h)$ of this approximation defined by

$$\mathcal{E}(x, h) = f'(x) - \frac{f(x+h) - f(x-h)}{2h}$$

- ii. Evaluate the accuracy $\mathcal{E}(h)$ of this approximation defined by

$$\mathcal{E}(h) = \sup_{x \in [0, 1]} |\mathcal{E}(x, h)|$$

(Hint: Find $s \in \mathbf{R}$ such that $\mathcal{E}(h) = O(h^s)$)

- iii. Conclude.

- b. The function f is perturbed as follows

$$f_n^\delta(x) = f(x) + \delta \sin\left(\frac{nx}{\delta}\right)$$

where $\delta \in]0, 1[$ and $n \in \mathbf{N}$; $n \geq 2$.

- iv. Find the error $\mathcal{E}_n^\delta(x, h)$ defined by

$$\mathcal{E}_n^\delta(x, h) = f'(x) - \frac{f_n^\delta(x+h) - f_n^\delta(x-h)}{2h}$$

- v. Evaluate the error $\mathcal{E}_n^\delta(h)$ defined by

$$\mathcal{E}_n^\delta(h) = \sup_{x \in [0, 1]} |\mathcal{E}_n^\delta(x, h)|$$

(Hint: Find $p \in \mathbf{R}$ and $q \in \mathbf{R}$ such that $\mathcal{E}_n^\delta(h) \leq O(h^p) + \delta O(h^q)$)

- vi. For a given perturbation level δ , describe the behavior of the perturbed error $\mathcal{E}_n^\delta(h)$ as the step size h tends to zero.
- vii. Assume that δ is known. What could be a “good” choice of the step size h to minimize the perturbed error $\mathcal{E}_n^\delta(h)$?