December 6, 2009 MATH 680A

Homework Assignment 4

1. Let A be a linear bounded operator on a Banach space. Show that if A is *degenerate*, i. e. the range of A is finite-dimensional then A is compact.

2. Show that if A_n is a sequence of compact operators on a Banach space such that $A_n \to A$ in the operator norm then A is compact (the set of compact operators is thereby closed in the strong operator topology).

3. Consider the integral operator

$$Tu(x) = \int_a^b K(x,y)\,u(y)\,dy \quad x \in [a,b].$$

Suppose that K is a degenerate kernel, such that

$$K(x,y) = \sum_{j=1}^{N} f_j(x)\,g_j(y)$$

where f_j and g_j are continuous on $[a,b]$. Find the eigenvalues and eigenfunctions of the operator T. [Consider first the case $N = 1$.]

4. Show that the right shift operator

$$R(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$$

defined on sequences $(a_0, a_1, \dots) \in \ell^2$ has no eigenvalues. [However, as we showed in class, its spectrum is the unit disk $\{|z| \leq 1\}$.]

5. Show that the spectrum of the shift operators R and L, defined as above and by

$$L(a_0, a_1, \dots) = (a_1, a_2, \dots)$$

on the spaces ℓ^p, $1 \leq p \leq \infty$, consists of all points of the unit disk.

6. Let A be a linear compact symmetric operator on a Hilbert space, such that $\langle Ax, x\rangle > 0$, $x \neq 0$ (this guarantees that all eigenvalues of A are positive). Show that the eigenvalues of A satisfy the following min-max principles:

$$\lambda_n = \max_{S_n} \min_{x \in S_n} \frac{\langle Ax, x\rangle}{\|x\|^2} \tag{a}$$

where the max is taken over all subspaces S_n that have dimension n and the min is over all $x \neq 0$ in that subspace. Furthermore,

$$\lambda_n = \min_{S_{n-1}} \max_{x \in S_{n-1}} \frac{\langle Ax, x \rangle}{\|x\|^2}. \tag{b}$$

(These relations are known as Fischer's and Courant's min-max principles, respectively.)