November 13, 2009 MATH 680A

Homework Assignment 3

Due Mon. Nov. 30, 2009, in class.

1. Show that every closed convex set K in a Hilbert space has a unique element of minimum norm. [Adapt the proof of the Lemma on Orthogonal Projection which addresses the case when K is a affine.]

2. If $f \in L^p \cap L^\infty$ then

   (a) $f \in L^q$ for all $q > p$ and $\|f\|_q \le \|f\|_\infty^{1-\lambda} \|f\|_p^\lambda$, where $\lambda = \frac{p}{q}$.

   (b) $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

3. Show that $L^\infty(\mathbb{R}) \ne \bigcap_{p\ge 1} L^p(\mathbb{R})$ (give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is in L^p for all $p \ge 1$ but not in L^∞.

4. Suppose $0 < p_0 < p_1 \le \infty$. Find examples of functions f on $(0, \infty)$ (with Lebesgue measure) such that $f \in L^p$ if and only if (a) $p_0 < p < p_1$ (b) $p_0 \le p \le p_1$ (c) $p = p_0$. [Look at the example of the functions $f(x) = x^{-a} |\log x|^b$.]

5. If $0 < p < 1$, the formula $\rho(f, g) = \int |f - g|^p \, d\mu$ defines a metric on L^p that makes L^p into a complete metric space. (To show completeness one can adapt the proof we did in class for $p \ge 1$).

6. (a) If X is a linear space with an inner product, show that the parallelogram law holds:
   $$\|x + y\|^2 + \|x - y\|^2 = \|x\|^2 + \|y\|^2, \quad x, y \in X.$$

   (b) Show that the norm on L^p does not come from an inner product (except in the trivial case when $\dim(L^p) \le 1$).

7. In this problem l^p, $p \in \{1, \infty\}$ denotes $L^p(\mathbb{N})$ with the counting measure. Define $\phi_n \in (l^\infty)^*$ by $\phi_n(f) = \frac{1}{n} \sum_{j=1}^{n} f(j)$. Then ϕ defined for each $f \in l^\infty$ by $\phi(f) = \lim_{n\to\infty} \phi_n(f)$ is an element of $(l^\infty)^*$ which does not come from any element of l^1.

8. If $g \in L^\infty(\mathbb{R}^n)$, the operator T defined by $Tf = fg$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. Its operator norm is at most $\|g\|_\infty$.