October 14, 2009 MATH 680A

Homework Assignment 2

Due Wed. Oct. 28, 2009, in class.

1. If X is a normed space, the closure of any subspace is a subspace.

2. Let X be a Banach space.

   (a) If $T \in \mathscr{L}(X,X)$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible; in fact, the series $\sum_0^\infty (I-T)^n$ converges in $\mathscr{L}(X,X)$ to T^{-1}.

   (b) If $T \in \mathscr{L}(X,X)$ is invertible and $\|S - T\| \le \|T^{-1}\|^{-1}$ then S is invertible. Thus the set of invertible operators is open in $\mathscr{L}(X,X)$.

3. If A is a nonempty set and $f : A \to [0, +\infty]$ is a nonnegative function we may define

$$\sum_{a\in A} f(\alpha) = \sup\Big\{ \sum_{a\in F} f(\alpha) : F \subset A,\ F\text{– finite}\Big\}$$

   (this coincides with the definition of a sum of a series if $A = \mathbb{N}$. We set

$$\ell^2(A) = \big\{ f : A \to \mathbb{C} : \sum_{a\in A} |f(\alpha)|^2 < +\infty \big\}.$$

   (which is a linear space) and define the inner product on $\ell^2(A)$ as $\langle f, g\rangle = \sum_{a\in A} f(\alpha)\,\overline{g(\alpha)}$ Show that $\ell^2(A)$ is a Hilbert space.

4. Show that every closed convex set in a Hilbert space has a unique element of minimal norm. (Hint: look at part (a) of the Theorem on Orthogonal Projection).

5. (See page 153 in Folland for definition of quotient space). Let X be a normed vector space and M a proper closed subspace of X.

   (a) $\|x + M\| = \inf\{\|x + y\| : y \in M\}$ is a norm on X/M.

   (b) For any $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \ge 1 - \varepsilon$

   (c) The projection map $\pi(x) = x + M$ from X to X/M has norm 1.

   (d) If X is complete, so is X/M. (Use the criterion of completeness using absolutely convergent series)

6. If $\|\cdot\|$ is a seminorm on the vector space X, let $M = \{x \in X : \|x\| = 0\}$. Then M is a subspace, and the map $x + M$ is a norm on X/M.

7. Let K be a convex set in a real linear space X, and let $p_K(x)$ denote the gauge function of K. Then $p_K(x) < 1$ if and only if x is an interior point of K. .