September 14, 2009 MATH 680A

Homework Assignment 1

Due Mon. Sep. 28, 2009, in class.

1. Given p, a prime number, define the p-adic distance for $x, y \in \mathbb{Q}$:

$$d_p(x,y) = p^{-v_p(x-y)}, \quad x \neq y; \quad d_p(x,y) = 0, \quad x = y,$$

   where $v_p(z) = v_p(r) - v_p(q)$ for $z = \pm r/q$, and $v_p(n)$, for n integer, is the exponent of p in the decomposition of n into prime numbers [cf. page 30 in Dieudonne.]

   Prove that $d_p(x,y)$ satisfies the strengthened form of the triangle inequality (the ultrametric property)

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}, \quad x, y, z \in \mathbb{Q}.$$

2. Problem 4, page 39.

3. Problem, page 35.

4. Problem 4, page 44.

5. Problem 1, page 55.

6. (a) Let φ be an increasing real-valued function defined on the interval $[0, \infty)$, and such that $\varphi(0) = 0$, $\varphi(u) > 0$ if $u > 0$ and $\varphi(u+v) \leq \varphi(u) + \varphi(v)$. Let $d(x,y)$ be a distance on a set E. Verify that $d_1(x,y) = \varphi(d(x,y))$ is another distance on E.

   (b) Prove that the functions

$$u^r \quad (0 < r \leq 1), \qquad \log(1+u), \qquad u/(1+u) \qquad \inf\{1, u\}$$

   satisfy the preceding conditions.

7. Show that the set $\mathbb{Q} \cap [0,1]$ violates the definition of a compact set: find an covering by open sets so that there is no finite subcovering.