MATH 680A December 6, 2009

Exam questions

1. Formulate and prove the contraction principle for maps of a metric space into itself. Obtain the Picard-Lindelöf's theorem as a corollary.
2. Formulate and prove the Arzela-Ascoli theorem. Describe the idea of the application to existence theorem for ODEs with continuous right-hand side (Peano's theorem).
3. Define the dual space of a normed vector space. Formulate and prove the (real space) Hahn-Banach theorem.
4. Prove Theorem 5.8 in Folland.
5. Formulate and prove the Uniform Boundedness Principle. Use it to prove that any weakly convergent sequence in a Banach space is bounded.
6. Formulate and prove the Lemma on Orthogonal Projection in a Hilbert space.
7. Formulate and prove the Riesz representation theorem
8. Define the gauge function for a convex set in a normed space and state its basic properties. Formulate and prove the hyperplane separation theorem for convex sets in a normed space.
9. Define the resolvent, resolvent set and spectrum for a bounded linear operator on a normed space. Show that the resolvent set is open, and the spectrum is compact.
10. Prove Bessel's inequality for an orthonormal set in a Hilbert space. Formulate and prove the Riesz-Fischer theorem (theorem 5.27 in Folland). Give an example of a complete orthonormal set in $L^2(0,1)$.
11. Prove the converse of Hölder's inequality for $1 \leqslant p \leqslant \infty$ in the case of measure space with finite measure.
12. Prove that a normed space is complete if and only of every absolutely convergent series converges. Use this fact to show that L^p, $1 \leqslant p < \infty$ is complete.
13. For a linear compact symmetric operator on a Hilbert space describe the properties of eigenvectors and eigenvalues. Prove that eigenvectors form a complete orthonormal set, that eigenvalues have finite multiplicity (except possibly zero eigenvalue) and may only accumulate at zero.
14. Formulate and prove the Fredholm alternative for a linear symmtric compact operator on a Hilbert space.