<u>Theorem 1</u> Suppose $U \subseteq \mathbb{C}$ is open, <u>bounded</u>. $\overline{U}$ – closure, ∂U – boundary.

Suppose $f_n : \overline{U} \to \mathbb{C}$ – continuous

$$f_n \rightrightarrows f \text{ on } U$$

Then $f_n \rightrightarrows \overline{f}$ on $\overline{U}$

where $\overline{f}(z) = f(z)\ z \in U$, $\overline{f}(z) = \lim_n f_n(z)$ $z \in \partial U$

Moreover, $\overline{f}$ is unif. cont. on $\overline{U}$.

<u>Proof.</u> <u>Step 1</u>: Since $\overline{U}$ is bounded and closed, it is compact.

$\forall n \in \mathbb{N}$ f_n is continuous on $\overline{U}$

$\Rightarrow$ f_n is uniformly continuous on $\overline{U}$.

<u>Show that f is unif. cont. on U.</u>

Let $\varepsilon > 0$ be given. $\forall z_1, z_2 \in U$

$$|f(z_1) - f(z_2)| \le |f(z_1) - f_n(z_1)| + |f_n(z_1) - f_n(z_2)| + |f_n(z_2) - f(z_2)|$$

Choose n so large that

$\forall z \in U$ $|f_n(z) - f(z)| < \frac{\varepsilon}{3}$.

for this n choose δ so small that $|z_1 - z_2| < \delta \Rightarrow |f_n(z_1) - f_n(z_2)| < \frac{\varepsilon}{3}$

Then $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$
$= \varepsilon.$

$\Rightarrow f$ is unif. cont. on U.

<u>Step 2</u>: <u>Show that $\forall z_0 \in \partial U$</u>
<u>$\lim_{n\to\infty} f_n(z_0)$ exists, and</u>
is equal to the limit of $f(z)$
as $z \to z_0$, $z \in U$.

Let $z_k \to z_0$, $z_k \in U$.

$|f_n(z_k) - f_m(z_\ell)| \le |f_n(z_k) - f(z_k)| +$

$|f(z_k) - f(z_\ell)| + |f(z_\ell) - f_m(z_\ell)|$

Given $\varepsilon > 0$.
Choose n, m so large that
$\forall z \in U \quad |f_n(z) - f(z)| < \frac{\varepsilon}{3}$

Choose k, ℓ so large that $|z_k - z_\ell| < \delta$
and $|f(z_k) - f(z_\ell)| < \frac{\varepsilon}{3}$.

Then $|f_n(z_k) - f_m(z_\ell)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

Thus $\forall k_n$ increasing on $\mathbb{N}$
the sequence $f_n(z_{k_n})$ is Cauchy
$\Rightarrow$ has a limit in $\mathbb{C}$;
moreover the limit does not depend on
the choice of the subsequence z_{k_n}.

Further, since $\forall n$ we can choose k_n so large that

$$\left| f_n(z_{k_n}) - f_n(z_0) \right| \le \frac{1}{n}$$

then $\lim_{n\to\infty} f_n(z_0)$ exists and is equal

to $\lim_{n\to\infty} f_n(z_{k_n})$ for this subsequence z_{k_n}, and therefore for any other as well.

<u>Step 3</u>: <u>Show that</u>

$$\sup_{z_0 \in \partial U} \left| f_n(z_0) - \bar{f}(z_0) \right| \xrightarrow[n\to\infty]{} 0.$$

Given $\varepsilon > 0$ $\forall z_0 \in \partial U$ choose z_{k_n}

such that

$$\left| f_n(z_0) - f_n(z_{k_n}) \right| < \frac{\varepsilon}{2}$$

and

$$\left| \bar{f}(z_0) - f_n(z_{k_n}) \right| < \frac{\varepsilon}{2}.$$

Then

$$\left| f_n(z_0) - \bar{f}(z_0) \right| \le \left| f_n(z_0) - f_n(z_{k_n}) \right| + \left| f_n(z_{k_n}) - \bar{f}(z_0) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since this is true for an arbitrary $z_0 \in \partial U$ the inequality $\le \varepsilon$ is true for the supremum.

Step 4: Since $f_n \rightrightarrows \bar{f}$ on $\bar{U}$
and each f_n is uniformly
continuous on $\bar{U}$
then by Step 1
$\bar{f}$ is uniformly continuous on $\bar{U}$.

Theorem 2: Suppose the series $\sum_{n=0}^{\infty} a_n z^n$
converges uniformly for $|z|<R$.
Then the same series must converge
uniformly for $|z|\le R$.

Proof: Let $U = \{z : |z|<R\}$;
$\bar{U} = \{z : |z| \le R\}$; $\partial U = \{z : |z| = R\}$.
$f_n(z) = \sum_{k=0}^{n} a_k z^k$ - partial sum of the series.
Each function $f_n(z)$ is continuous on $\bar{U}$
and $f_n(z) \rightrightarrows f(z)$ for $z \in U$.
By Theorem 1 $f_n(z) \rightrightarrows \bar{f}(z)$ for $z \in \bar{U}$
where

$$\bar{f}(z) = \begin{cases} f(z), & |z|<R \\ \lim_{n\to\infty} f_n(z), & |z| = R \end{cases}$$

(the limit exists and is equal the limit from the interior of $\bar{U}$, as in the proof of Thm. 1.)