Mon, Apr 22, 2019 **MATH 655, Midterm 2** Prof. V. Panferov

Name: (print) ______________________________

CSUN ID No. : Solutions.

This test contains 8 questions, on 8 pages. The perfects score is 44 points, the last question is a bonus worth an extra 6 points. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. No electronic devices; all cellphones must be turned off and put away for the duration of the test. Show all your work.

1. (6 points) If C is a closed simple contour such that $\{-1, 0, 1\} \not\subset C$ determine all possible values of the integral

$$\int_C \frac{z\,dz}{1-z^2}$$

depending on the location of C and its orientation. Give examples of contours corresponding to each value.

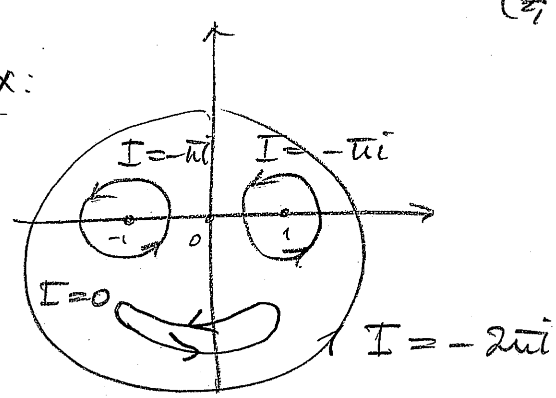
$f(z) = \frac{z}{1-z^2}$ has simple poles at $z = \pm 1$

$$f(z) = -\frac{z}{(z-1)(z+1)} = -\frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right)$$

$$\operatorname{Res}_{z=-1} f(z) = \operatorname{Res}_{z=1} f(z) = -\frac{1}{2}$$

$$I = \int_C f(z)\,dz = \pm 2\pi i \sum_{\substack{z=z_i \\ (z_i \text{ inside } C)}} \operatorname{Res}_{z=z_i} f(z) = 0, \pm\pi i, \pm 2\pi i$$

Ex:

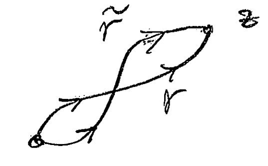


Changing the orientation reverses the sign of I.

2. (8 points) (a) Prove Morera's Theorem: If f is continuous in a region $U \subseteq \mathbb{C}$ and the contour integral $\int_C f(z)\,dz$ vanishes for every closed simple contour in U then f is analytic in U.

Given $z_0 \in U$

Define $F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ for $z \in U$.



The integral is path-independent since the difference between two paths can be decomposed into sum of integrals over simple closed contours, each of which is zero.

Also $F'(z) = \lim_{\xi \to z} \dfrac{F(\xi) - F(z)}{\xi - z} = \lim_{\xi \to z} \dfrac{1}{\xi - z} \int_{\xi}^{z} f(\xi)\,d\xi = f(z)$

(Since $\left| \dfrac{1}{\xi - z} \int_{\xi}^{z} f(\xi)\,d\xi - f(z) \right| \le \dfrac{1}{|\xi - z|} \int_{\xi}^{z} |f(\xi) - f(z)|\,d\xi < \varepsilon$ if $|\xi - z| < \delta$ ...)

Thus F is analytic on $U \Rightarrow F''$ exists on $U \Rightarrow f$ is analytic on U.

(b) Suppose that a series $\sum_{n=0}^{\infty} f_n(z)$, where each f_n is analytic over a region U, converges uniformly on U. Prove that the sum of the series $f(z)$ is analytic in U.

Take an open disk $D \subseteq U$ and a closed simple contour $C \subseteq D$.

Since $\forall n$ f_n is analytic on D, by Cauchy's Thm

$$\int_C f_n(z)\,dz = 0$$

Let $S_N(z) = \sum_{n=0}^{N} f_n(z)$. Since $S_N \rightrightarrows f$ on U then $S_N \rightrightarrows f$ on C.

Pass to the limit in the integral using uniform convergence:

$$\int_C f(z)\,dz = \int_C \sum_{n=0}^{\infty} f_n(z)\,dz = \sum_{n=0}^{\infty} \int_C f_n(z)\,dz = \sum_{n=0}^{\infty} 0 = 0$$

(exists since $f(z)$ is continuous on U)

Note: Used that D is simply connected.

Since C is arbitrary, $f(z)$ is analytic in D.

Continued...

Since D is arbitrary, $f(z)$ is analytic in U.

3. (6 points) Let $f(z)$ be an entire function, with $|f(z)| \le C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

Solution 1: Let $h(z) = \frac{f(z)}{z}$, $z \neq 0$

Then $|h(z)| \le C$ for $|z| < 1$

$\Rightarrow h(z)$ has a removable singularity at $z=0$

$$\Rightarrow \tilde{h}(z) = \begin{cases} h(z), & z \neq 0 \\ \lim_{z\to 0} h(z), & z = 0 \end{cases}$$ is entire, bounded on $\mathbb{C}$

By Liouville's Theorem $\tilde{h}(z) = A$ (a constant)

$\Rightarrow f(z) = \tilde{h}(z) z = Az$, $z \in \mathbb{C}$.

Solution 2 $\forall z_0 \in \mathbb{C}$

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{(\xi - z_0)^2} d\xi \right| \le \frac{1}{2\pi} \frac{C(R - |z_0|) 2\pi R}{R^2}$$

$C_R = \{z : |z - z_0| = R\}$ $\quad \le C(1+\varepsilon)$, $\varepsilon > 0$

for R large enough (dependent on z_0!)

Since ε is arbitrary

$|f'(z_0)| \le C$

Since f' is entire and bounded, $f'(z) = A$

$\Rightarrow f(z) = Az + B \Rightarrow f(z) = Az$.

Solution 3 $\forall z_0 \in \mathbb{C}$

$$|f''(z_0)| = \left| \frac{1}{\pi i} \int_{C_R} \frac{f(\xi)}{(\xi - z_0)^3} d\xi \right| \le \frac{1}{\pi} \frac{C(R - |z_0|) 2\pi R}{R^3} \xrightarrow[R \to \infty]{} 0$$

Therefore $|f''(z_0)| = 0$

$\Rightarrow f(z) = Az + B \Rightarrow f(z) = Az$.

Continued...

4. (6 points) Let $f(z)$ be an analytic function defined for $|z| \leq 1$ and let

$$u(x,y) = \mathrm{Re}\,(f(z)), \quad z = x + iy.$$

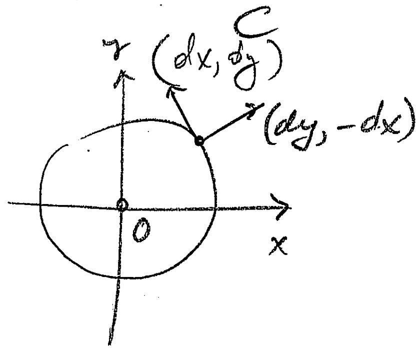
Prove that

$$\int_C u_y\,dx - u_x\,dy = 0,$$

where C is the positively oriented unit circle $x^2 + y^2 = 1$.

<u>Solution 1</u>: f is analytic $\Rightarrow$ u is harmonic inside $x^2+y^2=1$

$$\Rightarrow \int_C \frac{\partial u}{\partial n}\,ds = \iint_{x^2+y^2\leq 1} \Delta u \,dx\,dy = 0$$



on the unit circle C,

$$\frac{\partial u}{\partial n}\cdot ds = u_x\,dy - u_y\,dx$$

$$\Rightarrow \int_C u_y\,dx - u_x\,dy = -\int \frac{\partial u}{\partial n}\,ds = 0.$$

<u>Solution 2</u>: $f(z) = u + iv$ analytic $\Rightarrow$ $f'(z) = u_x + i v_x = u_x - i u_y$ analytic.

By Cauchy's Theorem $\int_C f'(z)\,dz = 0$

$$\Rightarrow \int_C (u_x - iu_y)(dx + i\,dy) = \int_C u_x\,dx + u_y\,dy + i\int_C -u_y\,dx + u_x\,dy = 0$$

$$\Rightarrow \int_C -u_y\,dx + u_x\,dy = 0.$$

Continued...

Solution 3:

$$f(z) = u + iv \quad \text{— analytic}$$

$$u_x = v_y, \quad u_y = -v_x$$

$$\int_C u_y\,dx - u_x\,dy = \int_C -v_x\,dx - v_y\,dy$$

$$= -\int_C v_x\,dx + v_y\,dy = -\int_a^b \left(v_x\,x'(t) + v_y\,y'(t)\right)dt$$

$$= -\int_a^b \frac{d}{dt}\,v(x(t), y(t))\,dt =$$

$$= v(x(a), y(a)) - v(x(b), y(b)) = 0$$

Since C is a closed contour $\Rightarrow$

$$x(a) = x(b), \quad y(a) = y(b).$$

5. (6 points) Evaluate the integrals:

(a) $\displaystyle\int_{|z-1|=1} \frac{ze^z}{(z-1)^3}\, dz$ for the contour oriented counterclockwise;

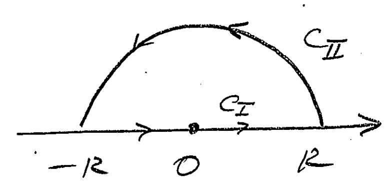
$$\int_{|z-1|=1} \frac{ze^z}{(z-1)^3}\, dz = 2\pi i \operatorname*{Res}_{z=1} \frac{ze^z}{(z-1)^3}$$

$$= 2\pi i \cdot \frac{1}{2!} \left(ze^z \right)'' \Big|_{z=1}$$

$$= \pi i \; (z+2) e^z \Big|_{z=1}$$

$$= 3e\pi i$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin 5x}{1+x^2}\, dx$, using the method of residues.



Calculate $\displaystyle\int_C \frac{z e^{5iz}}{1+z^2}\, dz$, $C = C_I + C_{II}$:

$$= 2\pi i \operatorname*{Res}_{z=i} \frac{ze^{5iz}}{1+z^2} = 2\pi i \left. \frac{ze^{5iz}}{z+i} \right|_{z=i} = 2\pi i \frac{ie^{-5}}{2i}$$

(the only pole inside C is $z=i$)

$$= \pi i e^{-5}.$$

Since $\displaystyle\left| \frac{z}{1+z^2} \right| \le \frac{R}{R^2-1} \xrightarrow[R\to\infty]{} 0$ for $|z|=R$

Jordan's Lemma $\Rightarrow$ $\displaystyle\int_{C_{II}} \frac{ze^{5iz}}{1+z^2}\, dz \xrightarrow[R\to\infty]{} 0$

Then $\displaystyle\int_C \frac{ze^{5iz}}{1+z^2}\, dz = \lim_{R\to\infty} \int_{C_I} \frac{ze^{5iz}}{1+z^2}\, dz$

$$= \int_{-\infty}^{\infty} \frac{ze^{5iz}}{1+z^2}\, dz = \pi i e^{-5}$$

By taking the imaginary part,

$$\int_{-\infty}^{\infty} \frac{x \sin 5x}{1+x^2}\, dx = \pi e^{-5}.$$

Continued...

6. (6 points) Determine all singularities of the functions in $\mathbb{C}\cup\{\infty\}$, classify them, and find principal parts of Laurent series about each singularity that admits a Laurent expansion:

(a) $\dfrac{e^{iz}-1-iz}{iz^3} = \dfrac{\frac{1}{2!}(iz)^2+\frac{1}{3!}(iz)^3+\cdots}{-(iz)^3} = -\dfrac{1}{2i}\dfrac{1}{z}-\dfrac{1}{6}-\cdots$

$z=0$ is a simple pole; $P_0=-\frac{1}{2i}\frac{1}{z}$ - principal part.

$z=\infty$ is an essential singularity

$$f(z)=-\frac{1}{2i}\frac{1}{z}-\frac{1}{6}-\sum_{k=1}^{\infty}\frac{1}{(k+3)!}(iz)^k$$

$$\Rightarrow P_\infty=-\sum_{k=1}^{\infty}\frac{1}{(k+3)!}(iz)^k \text{ - principal part at } z=\infty.$$

(b) $\dfrac{z^{1/3}-1}{z-1}$.

$z=0$ - branch point; no Laurent expansion.

($z^{1/3}=|z|^{1/3}e^{i\arg(z)/3}e^{2\pi i n/3}$, $n=0,1,2$ need a cut through $z=0$ to obtain a single-valued branch.)

$z=\infty$ - branch point; no Laurent expansion:

($f\left(\frac{1}{z}\right)=\dfrac{\frac{1}{z^{1/3}}-1}{\frac{1}{z}-1}$; $z^{-1/3}$ has a branch point at $z=0$)

$z=1$ - removable singularity

$$\frac{z^{1/3}-1}{z-1}=\frac{1}{z^{2/3}+z^{1/3}+1}\longrightarrow\frac{1}{1+e^{2\pi i n/3}+e^{4\pi i n/3}}$$

$n=0,1,2$ (depending on the branch)

principal part $P_1=0$.

Continued...

7. (6 points) Given the function element (f, D)

$$f(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots, \quad D = \{z : |z| < 1\}$$

find the Taylor series for its analytic continuation into the region $E = \{z : |z - i| < 1\}$. Is this analytic continuation unique?

$$f(z) = 1 - \ln_p(1-z)$$

Since $\ln_p(1-z) = (-z) - \frac{1}{2}(-z)^2 + \frac{1}{3}(-z)^3 - \frac{1}{4}(-z)^4 + \ldots$

$$= -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots$$

E has non-empty intersection with D

E, D are open $\Rightarrow$ $D \cap E$ is open, non-empty contains limit points of both D and E.

By local Uniqueness Theorem the analytic continuation is unique.

It must be given by the Taylor expansion of $1 - \ln_p(1-z)$ on E:

$$-\ln_p(1-z) = -\ln_p(1-i-(z-i)) = -\ln(1-i) - \ln_p\left(1 - \frac{z-i}{1-i}\right)$$

where $-\ln(1-i) = -\ln\sqrt{2}e^{-i\pi/4} = -\frac{1}{2}\ln 2 + \frac{i\pi}{4}$

is the principal value, to stay on the same branch.

$$\Rightarrow f(z) = 1 - \frac{1}{2}\ln 2 + \frac{i\pi}{4} + \frac{z-i}{1-i} + \frac{1}{2}\left(\frac{z-i}{1-i}\right)^2 + \ldots$$

8. (bonus: 6 points) Let $\varepsilon > 0$, and $D_\varepsilon = \{z : 0 < |z| < \varepsilon,\ 1/z \neq \pi n,\ n \in \mathbb{Z}\}$ and $f(z) = \csc\frac{1}{z}$. Show that $f(D_\varepsilon)$ is dense in $\mathbb{C}$.

Let $w \in \mathbb{C}$.

WTS: $|f(z) - w|$ is as small as we wish, for some $z \in D_\varepsilon$.

Suppose not; then $h(z) = \dfrac{1}{f(z) - w}$ satisfies

$$|h(z)| \le C_0 \text{ for } z \in D_\varepsilon.$$

At each $z_n = \dfrac{1}{\pi n}$, $n = \pm 1, \pm 2, \dots$

$h(z)$ has a removable singularity, and $h(z_n) = 0$ by continuity.

Then $h(z)$ extends analytically to $0 < |z| < \varepsilon$

$\Rightarrow h(z)$ has a removable singularity at $z = 0$.

Therefore

$$h(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad |z| < \varepsilon$$

and $h(z_n) = 0$, $n = \pm 1, \pm 2, \dots$, $z_n \to 0$.

By Identity Theorem for power series $h(z)$ must be identically $= 0$ for $|z| < \varepsilon$, a contradiction.

The end.

<u>Note</u>: The Casorati–Weierstrass Theorem does not apply literally, since $z_0 = 0$ is not an isolated singularity, but the idea of the proof carries over.