Hints for HW 3

March 5, 2008

2. (b) For the coercivity we need to prove the estimate

$$a(v,v) = \int_0^1 {v''}^2\,dx \geq \alpha(\|v\|^2 + \|v'\|^2 + \|v''\|^2), \quad \forall\, v \in C^2[0,1], \quad v(0) = v(1) = 0.$$

First notice that by Poincaré, $\|v\|^2 \leqslant \|v'\|^2$, since $v(0) = v(1) = 0$. Further, since $v(x)$ is smooth, and $v(0) = v(1) = 0$, we have $v'(x_0) = 0$ for some $0 < x_0 < 1$ (Right?). But then

$$v'(x) = \int_{x_0}^x v''(y)\,dy$$

so

$$|v'(x)| \leqslant \int_{x_0}^x |v''(y)|\,dy \leqslant \int_0^1 |v''(y)|\,dy \leqslant \|v''\|.$$

Taking the square and integrating over $x \in (0,1)$ we get $\|v'\|^2 \leqslant \|v''\|^2$. Putting the pieces together we get the coercivity estimate with $\alpha = \frac{1}{3}$.

6. I will prove the equivalent inequality

$$\|u - \int_\Omega u\,dx\| \leqslant C\|\nabla u\|,$$

where $\Omega = (0,1)^2$ is the unit square in $\mathbb{R}^2$. (You will have to show that it is indeed equivalent.)

First, consider the one-dimensional version,

$$\left|u(x) - \int_0^1 u(x_0)\,dx_0\right| \leqslant \int_0^1 |u'(x_0)|\,dx_0, \quad \forall\, x \in (0,1).$$

By the fundamental theorem of calculus,

$$u(x) = u(x_0) + \int_{x_0}^x u'(y)\,dy,$$

for any pair of numbers x, $x_0 \in (0,1)$. Therefore,

$$u(x_0) - \int_{x_0}^x |u'(y)|\,dy \leqslant u(x) \leqslant u(x_0) + \int_{x_0}^x |u'(y)|\,dy,$$

or, since $\int_{x_0}^{x} |u'(y)|\,dy \leqslant \int_0^1 |u'(y)|\,dy$,

$$u(x_0) - \int_0^1 |u'(y)|\,dy \leqslant u(x) \leqslant u(x_0) + \int_0^1 |u'(y)|\,dy$$

Since x_0 is arbitrary we may integrate over $x_0 \in (0,1)$;

$$\int_0^1 u(x_0)\,dx_0 - \int_0^1 |u'(y)|\,dy \leqslant u(x) \leqslant \int_0^1 u(x_0)\,dx_0 + \int_0^1 |u'(y)|\,dy.$$

Subtracting $\int_0^1 u(x_0)\,dx_0$ we obtain

$$\left|u(x) - \int_0^1 u(x_0)\,dx_0\right| \leqslant \int_0^1 |u'(y)|\,dy.$$

Now, in two dimensions, we have

$$u(x,y) = \int_0^1 u(x,y_0)\,dy_0 + \varepsilon_1(x,y)$$

where $|\varepsilon_1(x,y)| \leqslant \int_0^1 |u_y(x,y_0)|\,dy_0$ and

$$u(x,y_0) = \int_0^1 u(x_0,y_0)\,dx_0 + \varepsilon_2(x,y_0)$$

where $|\varepsilon_2(x,y_0)| \leqslant \int_0^1 |u_x(x_0,y_0)|\,dx_0$. Thus,

$$u(x,y) = \int_0^1 \int_0^1 u(x_0,y_0)\,dx_0\,dy_0 + \varepsilon_1(x,y) + \int_0^1 \varepsilon_2(x,y_0)\,dy_0$$

and

$$\begin{aligned}\left|u(x,y) - \int_0^1 \int_0^1 u(x_0,y_0)\,dx_0\,dy_0\right|^2 &\leqslant 2\,\varepsilon_1(x,y)^2 + 2\Big(\int_0^1 \varepsilon_2(x,y_0)\,dy_0\Big)^2 \\ \leqslant 2\int_0^1 |u_y(x,y_0)|^2\,dy_0 &+ 2\int_0^1 \int_0^1 |u_x(x_0,y_0)|^2\,dx_0\,dy_0.\end{aligned}$$

(We used the Cauchy-Schwarz.) Integrating over (x,y) we obtain

$$\left\|u - \int_0^1 \int_0^1 u(x_0,y_0)\,dx_0\,dy_0\right\|^2 \leqslant 2\|\nabla u\|^2,$$

which is the desired inequality, with $C = \sqrt{2}$.