May 1, 2008 MATH 592A

Homework Assignment 6

1. Consder the 1-D wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$

(a) Show that the change of variables

$$\xi = x - ct, \qquad \eta = x + ct$$

reduces the equation to the form

$$u_{\xi\eta} = 0.$$

(b) Show that the general solution of the above equation is

$$u(\xi, \eta) = f(\xi) + g(\eta),$$

where f and g are arbitrary functions, and so, the general solution of the 1-D wave equation is

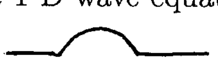
$$u(x,t) = f(x - ct) + g(x + ct).$$

(c) Consider the 1-D wave equation with the initial conditions $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$. Show that in that case the functions f and g can be determined from the system

$$f(x) + g(x) = \varphi(x)$$
$$-f'(x) + g'(x) = \frac{1}{c}\psi(x).$$

(d) Solve this system to obtain D'Alembert's formula,

$$u(x,t) = \frac{\varphi(x - ct) + \varphi(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy.$$

2. Solve the 'movie problem' (give snapshots of the solution for several different values of t) for the 1-D wave equation of the whole real line, with $\varphi = 0$ and ψ the function of the form  .

3. Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} = 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}_+ \\ u_x(0,t) = 0, \quad t \in \mathbb{R}_+, \end{aligned}$$

where $\mathbb{R}_+ = (0,\infty)$, $\psi = 0$ and $\varphi(x) = \mathbb{1}_{[1,2]}$. Draw the diagram of the solution in the (x,t)-plane. *Hint: extend the solution and the data to the whole real line as even functions.*

4. Solve Problem 11.6 in the book.

5. Solve Problem 11.7 in the book.

6. Solve Problem 11.8 in the book

7. Solve Problem 11.11 in the book

8. Solve Problem 11.13 in the book

9. Solve Problem 11.15 in the book.