Homework Assignment 5

Due on Tuesday, May 6, 2008, in class.

1. Let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basis in $L^2(\Omega)$. Prove that if

$$f = \sum_{n=1}^{\infty} c_n \varphi_n$$

(the series converges in the sense of L^2) then $c_n = (f, \varphi_n)$. *Hint: you need to justify the usual formal argument that the series may be multiplied term-by-term by a basis function* φ_m*...*

2. Fourier series on $(0, 2\pi)$.

   (a) Under what conditions on the coefficients c_n does the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ represent a real-valued function?

   (b) If $f \in L^2(0, 2\pi)$ then we have both

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

   Find the relation between c_n and a_n, b_n.

3. Solve Problem 6.1 from the book.

4. Solve Problem 6.2 from the book.

5. Solve Problem 8.3 from the book.

6. Energy method for the heat equation

   (a) Let u be a smooth solution of

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0, T) \\ u &= 0 \qquad \text{on} \quad \Gamma \times (0, T). \end{aligned}$$

   Show that for any $\Phi : \mathbb{R} \to \mathbb{R}$, smooth, convex, and with $\Phi'(0) = 0$

$$\frac{d}{dt} \int_{\Omega} \Phi(u(x, t))\, dx \leq 0.$$

   [Hint: Multiply the heat equation by $\Phi'(u)$ and integrate by parts.]

(b) Use part (a) to deduce the uniqueness of solutions of the problem

$$\begin{aligned} &u_t - \Delta u = f && \text{in} \quad \Omega \times (0,T) \\ &u = \varphi(x,t) && \text{on} \quad \Gamma \times (0,T) \\ &u(x,0) = g(x) && \text{in} \quad \Omega. \end{aligned}$$

(c) Show that $\forall T > 0$,

$$\int_0^T \|\nabla u\|^2 \, dt \leq \frac{1}{2}\|u_0\|^2.$$

[Hint: use problem 1(a) with $\Phi(u) = u^2$.]

(d) Show that

$$\frac{d}{dt}(t\|\nabla u\|^2) + 2t\|\Delta u\|^2 = \|\nabla u\|^2.$$

[Hint: Multiply the heat equation by $-t\Delta u$ and integrate by parts.]

(e) Use parts (a) and (b) to show the estimate

$$\|\nabla u\| \leq \frac{1}{\sqrt{2t}}\|u_0\|.$$

7. Solve <u>one</u> of the following

(a) Does there exist a unique solution of the Cauchy problem

$$x(x^2 + y^2)\, u_x + y^3\, u_y = 0, \quad u|_{y=0} = 1$$

in a neighborhood of the point $(x_0, 0)$ on the x-axis?

(b) Solve by the method of characteristics:

$$(1 + x^2)u_x + u_y = 0, \quad u|_{y=0} = g(x).$$

In what region of the (x,y)-plane is the solution uniquely determined by the data $g(x)$?