March 13, 2008 MATH 592A

Homework Assignment 4

Due on Tuesday, April. 1, 2008, in class.

1. Solve
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega = \{x^2 + y^2 > R\} \\ u = g(\vartheta) & \text{on } \Gamma = \{x^2 + y^2 = R\} \end{cases}$$
where $g(\vartheta) = 1 + 3\sin\vartheta$, and ϑ is the polar angle (introduce the polar coordinates).

2. Show that the solution of
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega = \{x^2 + y^2 > R\} \\ u = g(\vartheta) & \text{on } \Gamma = \{x^2 + y^2 = R\} \\ u(x,y) \to 0, & \text{as } x^2 + y^2 \to \infty \end{cases}$$
is given by Poisson's formula
$$u(x,y) = \frac{1}{2\pi} \int_0^{2\pi} Q_R(r, \vartheta - \varphi)\, g(\vartheta)\, d\varphi,$$
where
$$Q_R(r,\vartheta) = \frac{r^2 - R^2}{R^2 - 2Rr\cos\vartheta + r^2}.$$

3. Solve Problem 3.10 in the book.

4. Solve Problem 3.12 in the book.

5. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Show that (smooth) solutions of
$$\Delta u = f \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} + \alpha u = g \quad \text{on } \Gamma,$$
with $\alpha > 0$ are unique. (Hint: use the energy estimate for the equation satisfied by the difference of two solutions.)

6. Find the eigenvalues and eigenfunctions of the problem
$$-u'' = \lambda u \quad \text{in } \Omega = (0,1)$$
$$u'(0) = 0, \quad u(1) = 0.$$
Compute the Rayleigh quotient $R(v) = \frac{\|v'\|^2}{\|v\|^2}$ for the trial function $v(x) = 1 - x^2$ and compare with the first eigenvalue.