February 25, 2008 MATH 592A

Homework Assignment 3

Due on Tuesday, Mar. 3, 2008, in class.

1. (Problem 2.6 in the book) Give variational formulations and prove the existence of solutions of

$$-u'' = f \quad \text{in} \quad \Omega = (0,1)$$

with the boundary conditions

(b) $u(0) = u'(1) = 0$,

(c) $-u'(0) + u(0) = u'(1) = 0$.

2. (Problem 2.7 in the book) Consider the "beam equation"

$$u^{(4)} = f \quad \text{in} \quad \Omega = (0,1)$$

(b) $u(0) = u''(0) = u(1) = u''(1) = 0$,

(c) $u(0) = u'(0) = u'(1) = u'''(1) = 0$.

Give variational formulations and investigate existence and uniqueness of solutions to these problems.

3. Prove that the function $\Phi(x) = \frac{1}{4\pi|x|}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$ satisfies

(a) $|\Phi_{x_i}(x)| \leq \frac{C}{|x|^2}$, $|\Phi_{x_i x_j}(x)| \leq \frac{C}{|x|^3}$, $i, j = 1, 2, 3$.

(b) $(\Phi, \Delta\varphi) = -\varphi(0), \ \forall \varphi \in C_0^\infty(\mathbb{R}^3)$.

(The function $\Phi(x)$ is the fundamental solution of Laplace's equation on $\mathbb{R}^3$.)

4. Does the maximum principle hold for smooth solutions of

$$\Delta u + cu = 0, \quad \text{in } \Omega,$$

with $c > 0$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain? If yes, give a proof; if no give a counterexample.

5. (Problem 3.4 from the book) Prove Friedrichs' inequality

$$\|v\|_{L^2(\Omega)} \leq C\Big(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)}\Big)^{1/2}, \quad \text{for} \quad v \in C^1(\overline{\Omega}),$$

where Ω is a bounded domain in $\mathbb{R}^n$ with (smooth) boundary Γ. Hint: Integrate by parts in the identity $\int_\Omega v^2\,dx = \int_\Omega v^2 \Delta\varphi\,dx$, where $\varphi(x) = \frac{1}{2n}|x|^2$.

6. (Problem 3.5 from the book) Prove

$$\|v\| \leq C\Big(\|\nabla v\|^2 + \Big(\int_\Omega v\,dx\Big)^2\Big)^{1/2}, \quad \text{for} \quad v \in C^1(\overline{\Omega}),$$

where Ω is the unit square in $\mathbb{R}^2$.