Homework Assignment 2

Due Feb. 14, 2008, in class.

1. (Problem 2.2 in the book) Determine Green’s functions for the following problems,
   
   (a) \(-u'' = f\) in \(\Omega = (0, 1)\), with \(u(0) = u(1) = 0\)
   
   (b) \(-u'' + cu = f\) in \(\Omega = (0, 1)\), with \(u(0) = u(1) = 0\)

2. (Problem 2.3 in the book) Consider the nonlinear boundary-value problem

   \[-u'' + u = e^u, \quad \text{in } \Omega = (0, 1), \quad \text{with } u(0) = u(1) = 0.\]

   Use the maximum principle to show that all solutions are nonnegative, i.e. \(u(x) \geq 0\) for all \(x \in \bar{\Omega}\). Use the strong version of the maximum principle to show that all solutions are positive, i.e. \(u(x) > 0\) for all \(x \in \Omega\).

3. Prove the following

   (a) If \(f \in H^1(0, 1)\) then there exists \(\tilde{f} \in C[0, 1]\) such that \(f(x) = \tilde{f}(x)\) except for \(x\) in a set of measure zero.

   (b) If \(f^n \to f\) (as \(n \to \infty\)) in \(H^1(0, 1)\) then \(\tilde{f}^n \to \tilde{f}\) (as \(n \to \infty\)) uniformly on \([0, 1]\).

   (c) If \(f \in H^1_0(0, 1)\) then \(\tilde{f}(0) = \tilde{f}(1) = 0\).

4. (Problem A.2 from the book) Prove that if \(L : V \to \mathbb{R}\) is a bounded linear functional and \(v_i \in V\) is a sequence with \(L(v_i) = 0\) that converges to \(v \in V\) then \(L(v) = 0\).

5. (Problem A.4 from the book) Given that \(L^2(0, 1)\) is complete prove that \(H^1(0, 1)\) is complete. Hint: Assume that \(\|v_j - v_i\| \to 0\) as \(i, j \to \infty\). Show that there are \(v, w\) such that \(\|v_j - v\| \to 0, \|v_j' - w\| \to 0\), and that \(w = v'\) in the sense of weak derivative.

6. (Problem A.5 from the book) Let \(\Omega = (-1, 1)\) and let \(v : \Omega \to \mathbb{R}\) be defined by \(v(x) = 1\) if \(x \in (-1, 0)\) and \(v(x) = 0\) if \(x \in (0, 1)\). Prove that \(v \in L^2(\Omega)\) and that \(v\) can be approximated arbitrarily well in the \(L^2\) norm by \(C^1\) functions.

7. (Problem A.7 from the book) Check if the function \(v(x) = \log(-\log|x|^2)\) belongs to \(H^1(\Omega)\) if \(\Omega = \{x \in \mathbb{R}^2 : |x| < \frac{1}{2}\}\). Are functions in \(H^1(\Omega)\) necessarily bounded and continuous?