February 7, 2008 MATH 592A

Homework Assignment 2

Due Feb. 14, 2008, in class.

1. (Problem 2.2 in the book) Determine Green's functions for the following problems,
   (a) $-u'' = f$ in $\Omega = (0,1)$, with $u(0) = u(1) = 0$
   (b) $-u'' + cu = f$ in $\Omega = (0,1)$, with $u(0) = u(1) = 0$

2. (Problem 2.3 in the book) Consider the nonlinear boundary-value problem

$$-u'' + u = e^u, \quad \text{in } \Omega = (0,1), \quad \text{with } u(0) = u(1) = 0.$$

   Use the maximum principle to show that all solutions are nonnegative, i. e. $u(x) \geq 0$ for all $x \in \bar{\Omega}$. Use the strong version of the maximum principle to show that all solutions are positive, i. e. $u(x) > 0$ for all $x \in \Omega$.

3. Prove the following
   (a) If $f \in H^1(0,1)$ then there exists $\tilde{f} \in C[0,1]$ such that $f(x) = \tilde{f}(x)$ except for x in a set of measure zero.
   (b) If $f^n \to f$ (as $n \to \infty$) in $H^1(0,1)$ then $\tilde{f}^n \to \tilde{f}$ (as $n \to \infty$) uniformly on $[0,1]$.
   (c) If $f \in H_0^1(0,1)$ then $\tilde{f}(0) = \tilde{f}(1) = 0$.

4. (Problem A.2 from the book) Prove that if $L : V \to \mathbb{R}$ is a bounded linear functional and $v_i \in V$ is a sequence with $L(v_i) = 0$ that converges to $v \in V$ then $L(v) = 0$.

5. (Problem A.4 from the book) Given that $L^2(0,1)$ is complete prove that $H^1(0,1)$ is complete. Hint: Assume that $\|v_j - v_i\| \to 0$ as $i, j \to \infty$. Show that there are v, w such that $\|v_j - v\| \to 0$, $\|v_j' - w\| \to 0$, and that $w = v'$ in the sense of weak derivative.

6. (Problem A.5 from the book) Let $\Omega = (-1,1)$ and let $v : \Omega \to \mathbb{R}$ be defined by $v(x) = 1$ if $x \in (-1,0)$ and $v(x) = 0$ if $x \in (0,1)$. Prove that $v \in L^2(\Omega)$ and that v can be approximated arbitrarily well in the L^2 norm by C^1 functions.

7. (Problem A.7 from the book) Check if the function $v(x) = \log(-\log|x|^2)$ belongs to $H^1(\Omega)$ if $\Omega = \{x \in \mathbb{R}^2 : |x| < \frac{1}{2}\}$. Are functions in $H^1(\Omega)$ necessarily bounded and continuous?