January 24, 2008 MATH 592A

Homework Assignment 1

Due Jan. 31, 2008, in class.

1. Prove that if $u(x,y)$ is a function such that $u_{xy}(x,y)$, $u_{yx}(x,y)$ are continuous in a neighborhood of a point (x_0,y_0), then $u_{xy}(x_0,y_0) = u_{yx}(x_0,y_0)$.

2. Let
$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}, \quad f(0,0) = 0.$$
Show that f_{xy} and f_{yx} are defined at the point $(0,0)$, but $f_{xy}(0,0) \neq f_{yx}(0,0)$.

3. Use the divergence theorem to prove the following (Green's identities)

   (a) $\int_\Omega u\,\Delta v\,dx = -\int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Gamma u\,\frac{\partial v}{\partial n_x}\,dS_x$

   (b) $\int_\Omega (v\,\Delta u - u\,\Delta v)\,dx = \int_\Gamma (v\,\frac{\partial u}{\partial n_x} - u\,\frac{\partial v}{\partial n_x})\,dS_x$,

   where $\frac{\partial u}{\partial n_x}$ are the directional derivatives in the direction of the (outward) normal n_x, i.e. $\frac{\partial u}{\partial n_x} = \nabla u\cdot n_x$.

4. Show that for any constant k,

   (a) the function $u(x,y) = e^{kx}\cos ky$ is a solution of Laplace's equation $u_{xx}+u_{yy}=0$.

   (b) the function $u(x,y) = e^{kx}\,e^{ky}$ is a solution of the wave equation $u_{xx}-u_{yy}=0$

   (c) the function $u(x,y) = (k/2)x^2 + (1-k)y^2/2$ is a solution of Poisson's equation $u_{xx}+u_{yy}=1$

5. (Problem 1.4 in the book) Introduce spherical coordinates (r,θ,ϕ) defined by $x_1 = r\sin\theta\cos\phi$, $x_1 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$. Assume that the function u does not depend on θ and ϕ, i.e. $u = u(r)$. Show that
$$\Delta u = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{du}{dr}\Big).$$

6. (Problem 1.6 in the book) Let $\Omega = \{x\in\mathbb{R}^3 : |x| < 1\}$. Determine an explicit solution of the boundary-value problem
$$-\Delta u + c^2 u = f \quad \text{in } \Omega, \quad \text{with } u = g \quad \text{on } \Gamma,$$
assuming spherical symmetry and that c, f and g are constants. That is, solve
$$-(r^2u'(r))' + c^2r^2u(r) = r^2 f \quad \text{for } r\in(0,1), \quad \text{with } u(1) = g, \quad u(0) \text{ finite.}$$
Hint: Set $v(r) = ru(r)$.