MATH 592A May 12, 2008

Exam questions, Revised version

1. Formulate and prove the maximum principle for solutions of second-order ODEs

$$-(au')' + bu' + cu = f.$$

   State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the maximum principle is valid.

2. Let $G(x, y)$ be Green's function for the problem

$$\begin{aligned} -(au')' + bu' + cu &= f \quad \text{in } \Omega = (0,1), \\ u(0) = 0, &\quad u(1) = 0. \end{aligned}$$

   Find an explicit representation for the solution of the boundary-value problem with the conditions $u(0) = u_0$, $u(1) = u_1$.

3. Formulate and prove the Riesz representation theorem.

4. State the Lax-Milgram theorem and use it to prove that the variational formulation of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}$$

   has a unique weak solution. Here Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary Γ.

5. Define what it means for a family $(\varphi_n)_{n=1}^{\infty}$ to be an orthonormal basis in a Hilbert space V. If $(\varphi_n)_{n=1}^{\infty}$ is an othonormal set show that the best approximation of an element $v \in V$ by the first N functions φ_n is given by $\sum_{n=1}^{N} c_n \varphi_n$, where c_n are the Fourier coefficients of v.

6. Assume that the problem

$$\begin{aligned} -\Delta \varphi &= \lambda \varphi \quad \text{in } \Omega \\ \varphi &= 0 \quad \text{on } \Gamma \end{aligned}$$

   has infinitely many eigenfunctions $(\varphi_n)_{n=1}^{\infty}$ and that the corresponding eigenvalues λ_n are positive and increase to infinity. Prove that $(\varphi_n)_{n=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$.

7. Derive the formula for the fundamental solution of the heat equation $U(x,t)$ and show formally that the solution of the Cauchy problem with the initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}^n$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} U(x-y,t)\, u_0(y)\, dy.$$

8. If $u_0(x)$, $x \in \mathbb{R}^n$, is bounded, continuous, show that the function

$$u(x,t) = \int_{\mathbb{R}^n} U(x-y,t)\, u_0(y)\, dy.$$

is a solution of the heat equation for $t > 0$ and satisfies

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0).$$

Here $U(x,t)$ is the fundamental solution of the heat equation on $\mathbb{R}^n$.

9. Formulate and prove the parabolic maximum principle for the solutions of

$$\begin{aligned} &u_t - \Delta u = f, \quad \text{on } \ \Omega \times \mathbb{R}_+ \\ &u = g \quad \text{on } \ \Gamma \times \mathbb{R}_+ \\ &u(\cdot, 0) = u_0 \quad \text{on } \ \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary Γ.

10. Solve the problem

$$\begin{aligned} &u_t - \Delta u = 0 \quad \text{in } \ \Omega \times \mathbb{R}_+ \\ &\varphi = 0 \quad \text{on } \ \Gamma \times \mathbb{R}_+ \\ &u(x, 0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

using separation of variables. Show that the solution satisfies

$$\|u(\cdot,t)\| \leqslant \|u_0\|.$$

11. Describe the method of characteristics for the solution of a first-order linear equation

$$\begin{aligned} &a(x,y)u_x + b(x,y)u_y = f(x,y) \\ &u(x,0) = g(x), \quad x \in \mathbb{R}. \end{aligned}$$

Under what conditions on the functions a, b does the problem have solutions in a neighborhood of the x-axis for any smooth f and g?

12. Prove that the total energy

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx$$

is constant on the solutions of

$$\begin{aligned} &u_{tt} - \Delta u = 0, \quad \Omega \times \mathbb{R} \\ &u = 0 \quad \text{on } \Gamma \times \mathbb{R}. \end{aligned}$$

13. Prove using energy estimates that if

$$\begin{aligned} &u_{tt} - \Delta u = 0, \quad \text{on } \mathbb{R}^n \times \mathbb{R} \\ &u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

and u_0, v_0 vanish in a ball $B(x_0, t_0) \subseteq \mathbb{R}^n$ then $u(x_0, t_0) = 0$. (Finite speed of propagation for the wave equation, Theorem 11.3)

14. For a symmetric hyperbolic system

$$\begin{aligned} &u_t + \sum_{j=1}^{n} A_j \, u_{x_j} + B \, u = f, \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ &u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n \end{aligned}$$

prove the energy estimate

$$\|u(t)\| \leqslant C_t \Big(\|u_0\| + \Big(\int_0^t \|f(s)\|^2 \, ds \Big)^{\frac{1}{2}} \Big).$$