Exam questions

1. Formulate and prove the maximum principle for solutions of second-order ODEs

$$-(au')' + bu' + cu = f.$$

   State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the maximum principle is valid.

2. Prove the stability estimate for the solutions of the boundary-value problem

$$\begin{aligned} -(au')' + bu' + cu = f, &\quad \text{in } \Omega = (0,1), \\ u(0) = u_0, &\quad u(1) = u_1, \end{aligned}$$

   in the form

$$\max_{\Omega} |u| \leqslant \max\{|u_0|, |u_1|\} + C \max_{\Omega} |f|.$$

   State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the result is valid. Use the stability estimate to prove uniqueness of solutions of the boundary-value problem.

3. For the boundary-value problem

$$\begin{aligned} -(au')' + cu = f &\quad \text{in } \Omega = (0,1), \\ u(0) = 0, &\quad u(1) = 0, \end{aligned}$$

   define the Green's function $G(x, y)$ and prove that the solution is represented as

$$u(x) = \int_0^1 G(x, y)\, f(y)\, dy.$$

   State the conditions on the coefficients $a(x)$, $c(x)$ under which the result is valid.

4. Let $G(x, y)$ be Green's function for the problem

$$\begin{aligned} -(au')' + bu' + cu = f &\quad \text{in } \Omega = (0,1), \\ u(0) = 0, &\quad u(1) = 0. \end{aligned}$$

   Find an explicit representation for the solution of the boundary-value problem with the conditions $u(0) = u_0$, $u(1) = u_1$.

5. Formulate and prove the lemma on the orthogonal projection in a Hilbert space.

6. Formulate and prove the Riesz representation theorem

7. Let $\Omega = (-1,1)^2 \subseteq \mathbb{R}^2$. Define the Sobolev space $H^1(\Omega)$. For which α is the function $f(x) = |x|^\alpha$ in $H^1(\Omega)$?

8. Formulate and prove Poincare's inequality for functions $v \in H_0^1(0,1)$.

9. Define the notion of weak solution of the boundary-value problem

$$
\begin{aligned}
-(au')' + bu' + cu &= f \quad \text{in } \Omega = (0,1), \\
u(0) = 0, \quad u(1) &= 0
\end{aligned}
$$

10. State the Lax-Milgram theorem and use it to prove that the variational formulation of

$$
\begin{aligned}
-(au')' + bu' + cu &= f \quad \text{in } \Omega = (0,1), \\
u(0) = 0, \quad u(1) &= 0
\end{aligned}
$$

has a unique weak solution. State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the result is valid.

11. Formulate and prove the Dirichlet principle for the weak solution of

$$
\begin{aligned}
-(au')' + cu &= f \quad \text{in } \Omega = (0,1), \\
u(0) = 0, \quad u(1) &= 0.
\end{aligned}
$$

(Theorem A.2). State the conditions on the coefficients $a(x)$, $c(x)$ under which the result is valid.

12. Define the fundamental solution of Laplace's equation on $\mathbb{R}^n$ and prove that it satisfies

$$(\Phi, \Delta\varphi) = -\varphi(0), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)$$

for $n = 2$.

13. Give a variational formulation of the boundary-value problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma
\end{aligned}
$$

and use Lax-Milgram's theorem to prove existence, uniqueness and continuous dependence on the data. Here Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary Γ.

14. Show that if

$$\lambda_1 = \min_{0 \neq v \in H_0^1(\Omega)} \frac{\|\nabla v\|^2}{\|v\|^2},$$

and the minimum is achieved for a certain function $\varphi_1 \in H_0^1$ then λ_1 and φ_1 are an eigenvalue and an eigenfunction of $-\Delta$ with the boundary conditions $\varphi = 0$ on Γ. (Theorem 6.2).

15. Define what it means for a family $(\varphi_n)_{n=1}^{\infty}$ to be an orthonormal basis in a Hilbert space V. If $(\varphi_n)_{n=1}^{\infty}$ is an othonormal set show that the best approximation of an element $v \in V$ by the first N functions φ_n is given by $\sum_{n=1}^{N} c_n \varphi_n$, where c_n are the Fourier coefficients of v.

16. Assume that the problem

$$
\begin{aligned}
-\Delta\varphi &= \lambda\varphi \quad \text{in } \Omega \\
\varphi &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

has infinitely many eigenfunctions $(\varphi_n)_{n=1}^{\infty}$ and that the corresponding eigenvalues λ_n are positive and increase to infinity. Prove that $(\varphi_n)_{n=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$.

17. Derive the formula for the fundamental solution of the heat equation $U(x,t)$ and show formally that the solution of the Cauchy problem with the initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}^n$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} U(x-y,t)\, u_0(y)\, dy.$$

18. If $u_0(x)$, $x \in \mathbb{R}^n$, is bounded, continuous, show that the function

$$u(x,t) = \int_{\mathbb{R}^n} U(x-y,t)\, u_0(y)\, dy.$$

is a solution of the heat equation for $t > 0$ and satisfies

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0).$$

Here $U(x,t)$ is the fundamental solution of the heat equation on $\mathbb{R}^n$.

19. Solve the problem

$$
\begin{aligned}
&u_t - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R}_+ \\
&\varphi = 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \\
&u(x,0) = u_0(x), \quad x \in \Omega,
\end{aligned}
$$

using separation of variables. Show that the solution satisfies

$$\|u(\cdot,t)\| \leqslant \|u_0\|.$$

20. Describe the method of characteristics for the solution of a first-order linear equation

$$\begin{aligned} &a(x,y)u_x + b(x,y)u_y = f(x,y) \\ &u(x,0) = g(x), \quad x \in \mathbb{R}. \end{aligned}$$

Under what conditions on the functions a, b does the problem have solutions in a neighborhood of the x-axis for any smooth f and g?

21. Derive D'Alembert's formula for the solution of

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad (x,t) \in \mathbb{R}^2 \\ &u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

22. Prove that the total energy

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx$$

is constant on the solutions of

$$\begin{aligned} &u_{tt} - \Delta u = 0, \quad \Omega \times \mathbb{R} \\ &u = 0 \quad \text{on } \Gamma \times \mathbb{R}. \end{aligned}$$

23. Prove using energy estimates that if

$$\begin{aligned} &u_{tt} - \Delta u = 0, \quad \text{on } \mathbb{R}^n \times \mathbb{R} \\ &u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

and u_0, v_0 vanish in a ball $B(x_0, t_0) \subseteq \mathbb{R}^n$ then $u(x_0, t_0) = 0$. (Finite speed of propagation for the wave equation, Theorem 11.3)

24. Define the general form of a symmetric hyperbolic system and state the theorem on the global existence of the Cauchy problem for such system in one spatial dimension. Indicate the method of proof.

25. For a symmetric hyperbolic system

$$\begin{aligned} &u_t + \sum_{j=1}^n A_j \, u_{x_j} + B \, u = f, \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ &u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n \end{aligned}$$

prove the energy estimate

$$\|u(t)\| \leqslant C_t \Big(\|u_0\| + \Big(\int_0^t \|f(s)\|^2 \, ds \Big)^{\frac{1}{2}} \Big).$$