Exam questions

1. Formulate and prove the maximum principle for solutions of second-order ODEs

\[-(au')' + bu' + cu = f.\]

State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the maximum principle is valid.

2. Prove the stability estimate for the solutions of the boundary-value problem

\[-(au')' + bu' + cu = f, \quad \text{in } \Omega = (0, 1),
\]

\[u(0) = u_0, \quad u(1) = u_1,\]

in the form

\[
\max_{\Omega} |u| \leq \max\{|u_0|, |u_1|\} + C \max_{\Omega} |f|.
\]

State the conditions on the coefficients $a(x)$, $b(x)$, $c(x)$ under which the result is valid. Use the stability estimate to prove uniqueness of solutions of the boundary-value problem.

3. For the boundary-value problem

\[-(au')' + cu = f \quad \text{in } \Omega = (0, 1),
\]

\[u(0) = 0, \quad u(1) = 0,\]

define the Green’s function $G(x, y)$ and prove that the solution is represented as

\[u(x) = \int_0^1 G(x, y) f(y) dy.\]

State the conditions on the coefficients $a(x)$, $c(x)$ under which the result is valid.

4. Let $G(x, y)$ be Green’s function for the problem

\[-(au')' + bu' + cu = f \quad \text{in } \Omega = (0, 1),
\]

\[u(0) = 0, \quad u(1) = 0.\]

Find an explicit representation for the solution of the boundary-value problem with the conditions $u(0) = u_0$, $u(1) = u_1$.

5. Formulate and prove the lemma on the orthogonal projection in a Hilbert space.
6. Formulate and prove the Riesz representation theorem

7. Let $\Omega = (-1, 1)^2 \subseteq \mathbb{R}^2$. Define the Sobolev space $H^1(\Omega)$. For which $\alpha$ is the function $f(x) = |x|^\alpha$ in $H^1(\Omega)$?

8. Formulate and prove Poincare’s inequality for functions $v \in H^1_0(0, 1)$.

9. Define the notion of weak solution of the boundary-value problem

$$-(au')' + bu' + cu = f \quad \text{in } \Omega = (0, 1),$$
$$u(0) = 0, \quad u(1) = 0$$

10. State the Lax-Milgram theorem and use it to prove that the variational formulation of

$$-(au')' + bu' + cu = f \quad \text{in } \Omega = (0, 1),$$
$$u(0) = 0, \quad u(1) = 0$$

has a unique weak solution. State the conditions on the coefficients $a(x), b(x), c(x)$ under which the result is valid.

11. Formulate and prove the Dirichlet principle for the weak solution of

$$-(au')' + cu = f \quad \text{in } \Omega = (0, 1),$$
$$u(0) = 0, \quad u(1) = 0.$$  
(Theorem A.2). State the conditions on the coefficients $a(x), c(x)$ under which the result is valid.

12. Define the fundamental solution of Laplace’s equation on $\mathbb{R}^n$ and prove that it satisfies

$$(\Phi, \Delta \varphi) = -\varphi(0), \quad \forall \varphi \in C^\infty_0(\mathbb{R}^n)$$

for $n = 2$.

13. Give a variational formulation of the boundary-value problem

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \Gamma$$

and use Lax-Milgram’s theorem to prove existence, uniqueness and continuous dependence on the data. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$, with smooth boundary $\Gamma$.

14. Show that if

$$\lambda_1 = \min_{\theta \neq v \in H^1_0(\Omega)} \frac{\|\nabla \varphi\|^2}{\|v\|^2},$$
and the minimum is achieved for a certain function \( \varphi_1 \in H_0^1 \) then \( \lambda_1 \) and \( \varphi_1 \) are an eigenvalue and an eigenfunction of \(-\Delta\) with the boundary conditions \( \varphi = 0 \) on \( \Gamma \).

(Theorem 6.2).

15. Define what it means for a family \((\varphi_n)_{n=1}^{\infty}\) to be an orthonormal basis in a Hilbert space \( V \). If \((\varphi_n)_{n=1}^{\infty}\) is an orthonormal set show that the best approximation of an element \( v \in V \) by the first \( N \) functions \( \varphi_n \) is given by \( \sum_{n=1}^{N} c_n \varphi_n \), where \( c_n \) are the Fourier coefficients of \( v \).

16. Assume that the problem

\[
-\Delta \varphi = \lambda \varphi \quad \text{in } \Omega \\
\varphi = 0 \quad \text{on } \Gamma
\]

has infinitely many eigenfunctions \((\varphi_n)_{n=1}^{\infty}\) and that the corresponding eigenvalues \( \lambda_n \) are positive and increase to infinity. Prove that \((\varphi_n)_{n=1}^{\infty}\) form an orthonormal basis of \( L^2(\Omega) \).

17. Derive the formula for the fundamental solution of the heat equation \( U(x, t) \) and show formally that the solution of the Cauchy problem with the initial data \( u(x, 0) = u_0(x) \), \( x \in \mathbb{R}^n \) is given by

\[
\int_{\mathbb{R}^n} U(x - y, t) u_0(y) \, dy.
\]

18. If \( u_0(x), x \in \mathbb{R}^n \), is bounded, continuous, show that the function

\[
\int_{\mathbb{R}^n} U(x - y, t) u_0(y) \, dy.
\]

is a solution of the heat equation for \( t > 0 \) and satisfies

\[
\lim_{(x,t) \to (x_0,0)} u(x, t) = u_0(x_0).
\]

Here \( U(x, t) \) is the fundamental solution of the heat equation on \( \mathbb{R}^n \).

19. Solve the problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in } \Omega \times \mathbb{R}^+ \\
\varphi &= 0 \quad \text{on } \Gamma \times \mathbb{R}^+ \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

using separation of variables. Show that the solution satisfies

\[
\|u(\cdot, t)\| \leq \|u_0\|.
\]
20. Describe the method of characteristics for the solution of a first-order linear equation

\[ a(x, y)u_x + b(x, y)u_y = f(x, y) \]
\[ u(x, 0) = g(x), \quad x \in \mathbb{R}. \]

Under what conditions on the functions \( a, b \) does the problem have solutions in a neighborhood of the \( x \)-axis for any smooth \( f \) and \( g \)?

21. Derive D’Alembert’s formula for the solution of

\[ u_{tt} - u_{xx} = 0, \quad (x, t) \in \mathbb{R}^2 \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \quad x \in \mathbb{R}. \]

22. Prove that the total energy

\[ \mathcal{E}(t) = \frac{1}{2} \int_{\Omega} u_t(x, t)^2 + |\nabla u(x, t)|^2 \, dx \]

is constant on the solutions of

\[ u_{tt} - \Delta u = 0, \quad \Omega \times \mathbb{R} \]
\[ u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}. \]

23. Prove using energy estimates that if

\[ u_{tt} - \Delta u = 0, \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \quad x \in \mathbb{R}^n, \]

and \( u_0, v_0 \) vanish in a ball \( B(x_0, t_0) \subseteq \mathbb{R}^n \) then \( u(x_0, t_0) = 0 \). (Finite speed of propagation for the wave equation, Theorem 11.3)

24. Define the general form of a symmetric hyperbolic system and state the theorem on the global existence of the Cauchy problem for such system in one spatial dimension. Indicate the method of proof.

25. For a symmetric hyperbolic system

\[ u_t + \sum_{j=1}^{n} A_j u_{x_j} + Bu = f, \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+ \]
\[ u(\cdot, 0) = u_0 \quad \text{in} \quad \mathbb{R}^n \]

prove the energy estimate

\[ \|u(t)\| \leq C_t \left( \|u_0\| + \left( \int_0^t \|f(s)\|^2 \, ds \right)^{\frac{1}{2}} \right). \]