Tue, Sept 18, 2012 **MATH 480** **Quiz 2**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Find the time-independent (steady state) solutions of the heat equation: $u_t - Ku_{xx} = 0$, $x \in (0, L)$, with the boundary conditions: $u_x(0,t) = h(T_0 - u(0,t))$, $u_x(L,t) = \Phi$, where h, T_0 and Φ are positive constants.

Time independent $\Rightarrow u_{xx} = 0 \Rightarrow u(x) = Ax + B$.

$u_x(0) = A = h(T_0 - B)$

$u_x(L) = A = \Phi$

$\Rightarrow A = \Phi, \quad B = T_0 - \frac{\Phi}{h}$

$u(x) = \Phi x + T_0 - \frac{\Phi}{h}$.

2. Find the separated solutions of the equation $u_{xx} + yu_y + u = 0$.

$u(x,y) = X(x)Y(y)$

$$\frac{X''}{X} + y\frac{Y'}{Y} + 1 = 0$$

$$\frac{X''}{X} = \lambda \; ; \; y\frac{Y'}{Y} + 1 = -\lambda$$

$$X(x) = \begin{cases} A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda}), & \lambda > 0 \\ Ax + B, & \lambda = 0 \\ A\cosh(x\sqrt{-\lambda}) + B\sinh(x\sqrt{-\lambda}), & \lambda < 0. \end{cases}$$

$$yY' + (1+\lambda)Y = 0$$

$$Y(y) = Cy^{-(1+\lambda)}$$

$$u(x,y) = \begin{cases} y^{-(1+\lambda)}(A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})), & \lambda > 0 \\ y^{-(1+\lambda)}(Ax + B), & \lambda = 0 \\ y^{-(1+\lambda)}(A\cosh(x\sqrt{-\lambda}) + B\sinh(x\sqrt{-\lambda})), & \lambda < 0. \end{cases}$$

Please turn over...

3. Find the separated solutions of the heat equation $u_t - u_{xx} = 0$ in the region $(x,t) \in (0,L) \times (0,\infty)$ that also satisfy the additional boundary conditions:

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad \text{for } t > 0.$$

$u(x,t) = X(x)\,Y(t)$.

$$\frac{Y'}{Y} - \frac{X''}{X} = 0$$

$\frac{Y'}{Y} = -\lambda$ $\quad$ $\frac{X''}{X} = -\lambda$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{cases}$$

$Y' + \lambda Y = 0$

$Y(t) = C e^{-\lambda t}$

$$X(x) = \begin{cases} A\cos x\sqrt{\lambda} + B\sin x\sqrt{\lambda}, & \lambda > 0 \\ Ax + B, & \lambda = 0 \\ A\cosh x\sqrt{-\lambda} + B\sinh x\sqrt{-\lambda}, & \lambda < 0 \end{cases}$$

$\lambda = 0$: $\quad A = 0, \quad X(x) = B$ (arbitrary)

$\lambda < 0$: $\quad A \cdot 0 + B \cdot 1 = 0 \Rightarrow B = 0$.

$A \cdot \underbrace{\sqrt{-\lambda}\sinh L\sqrt{-\lambda}}_{>0} = 0 \Rightarrow A = 0$

(only trivial solns.)

$\lambda > 0$: $\quad A \cdot 0 + B \cdot 1 = 0 \Rightarrow B = 0$.

$A\sqrt{\lambda}\sin L\sqrt{\lambda} = 0 \Rightarrow$ either $A = 0$ (trivial)

or $\sin L\sqrt{\lambda} = 0$

$\Leftrightarrow L\sqrt{\lambda} = \pi n, \quad n \in \mathbb{Z}$.

$\Rightarrow \lambda = \left(\frac{\pi n}{L}\right)^2, \quad n = 1, 2, \ldots$

$u(x,t) = A_0$ or $u(x,t) = A_n \cos\frac{\pi n x}{L} e^{-\left(\frac{\pi n}{L}\right)^2 t}$