Name: (print) Solutions.

This test includes 5 questions in the main part (64 points in total) and one bonus question (worth an extra 6 points), on 10 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (8 points) Consider the PDE $u_{xx} - u_{tt} = 0$.

   (a) Show that $u(x,t) = f(x+t) + g(x-t)$ is a solution for any f and g smooth.

$$u_x = f'(x+t) + g'(x-t)$$
$$u_{xx} = f''(x+t) + g''(x-t)$$
$$u_t = f'(x+t) - g'(x-t)$$
$$u_{tt} = f''(x+t) + g''(x-t)$$
$$\Rightarrow u_{xx} - u_{tt} = 0.$$

   (b) Show that $u(x,t) = \cos ax \sin at$ is a solution for any $a \in \mathbb{R}$.

$$u_x = -a \sin ax \sin at$$
$$u_{xx} = -a^2 \cos ax \sin at$$
$$u_t = a \cos ax \cos at$$
$$u_{tt} = -a^2 \cos ax \sin at$$
$$\Rightarrow u_{xx} - u_{tt} = 0.$$

2. (8 points) Find the t-independent (steady-state) solution of the equation $u_t - u_{xx} = u$, $0 < x < \pi$ with the boundary conditions $u(0,t) = 0$, $u_x(\pi, t) = 1$.

t- independent $\Rightarrow$ $u = u(x)$

$-u_{xx} = u$, $u_{xx} + u = 0$

$u(x) = A\cos x + B\sin x$

$u(0) = A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$

$u_x(x) = B\cos x$

$u_x(\pi) = B \cdot (-1) = -B = 1 \Rightarrow B = -1$

$u(x) = -\sin x$.

3. (16 points) Consider the PDE $u_{xx} - u_{tt} - u = 0$.

(a) Show that the ODEs satisfied by the functions $X(x)$ and $Y(t)$ in a complex separated solution $u(x,t) = X(x)\,Y(t)$ take the form $X'' - \lambda X = 0$, $Y'' - (\lambda - 1)Y = 0$, $\lambda \in \mathbb{C}$.

$u_{xx} - u_{tt} - u = X''Y - XY'' - XY = 0$

$$\underbrace{\frac{X''}{X}}_{=\lambda} - \underbrace{\frac{Y''}{Y} - 1}_{=-\lambda} = 0$$

$X'' = \lambda X \Rightarrow X'' - \lambda X = 0$

$-\frac{Y''}{Y} - 1 = -\lambda \Rightarrow \frac{Y''}{Y} + 1 = \lambda \Rightarrow Y'' - (\lambda - 1)Y = 0$.

(b) Show that <u>all</u> complex separated solutions for $\lambda \notin \{0,1\}$ are linear combinations of the functions e^{ax+bt} for some $a, b \in \mathbb{C}$. Find the relation satisfied by a and b.

If $\lambda \neq 0, 1$ then

$$X(x) = C_1 e^{\sqrt{\lambda}\, x} + C_2 e^{-\sqrt{\lambda}\, x}$$

$$Y(t) = C_3 e^{\sqrt{\lambda-1}\, t} + C_4 e^{-\sqrt{\lambda-1}\, t}$$

$$u(x,t) = X(x)Y(t) = A_1 e^{\sqrt{\lambda}x + \sqrt{\lambda-1}t} + A_2 e^{-\sqrt{\lambda}x + \sqrt{\lambda-1}t} + A_3 e^{\sqrt{\lambda}x - \sqrt{\lambda-1}t} + A_4 e^{-\sqrt{\lambda}x - \sqrt{\lambda-1}t}.$$

To find the relation, substitute e^{ax+bt} into the PDE $\Rightarrow (a^2-b^2-1)e^{ax+bt} = 0$

$$\Rightarrow a^2 - b^2 - 1 = 0.$$

(c) Find all complex separated solutions with $\lambda \notin \{0,1\}$ which are <u>bounded</u> for each $x \in \mathbb{R}$ in the form $|u(x,t)| \leq M_x$, for a certain constant M_x and all t, $-\infty < t < \infty$.

Since all such solutions are linear combinations of e^{ax+bt}, b must be purely imaginary, $b = i\omega$.

Then $b^2 = -\omega^2$, $a^2 = 1-\omega^2$,

$a = \pm\sqrt{1-\omega^2}$ is real if $|\omega| < 1$ and purely imaginary if $|\omega| > 1$.

$$u(x,t) = A_1 e^{\sqrt{1-\omega^2}x + i\omega t} + A_2 e^{-\sqrt{1-\omega^2}x + i\omega t} + A_3 e^{\sqrt{1-\omega^2}x - i\omega t} + A_4 e^{-\sqrt{1-\omega^2}x - i\omega t}$$

(d) Are there nontrivial (non-zero) separated solutions which are bounded by the same constant for <u>all</u> x and t? Are there solutions which satisfy $|u(x,0)| \to \infty$ as $x \to \pm\infty$?

If $a = \pm\sqrt{1-\omega^2}$ is purely imaginary ($|\omega| > 1$)

then every solution in part (c) is <u>bounded</u> for all $-\infty < x < \infty$, $-\infty < t < \infty$.

If $a = \pm\sqrt{1-\omega^2}$ is real ($|\omega| < 1$)

then these solutions are unbounded as $|x| \to \infty$.

4. (16 points – theoretical question) Let $(\varphi_n)_{n=1}^{\infty}$ be an orthogonal set of functions on (a,b) (such that $\|\varphi_n\| \neq 0$), and let f be a square-integrable function on (a,b).

(a) Write the definition of the Fourier coefficients $\hat{c}_n$ of f.

$$\hat{c}_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}.$$

(b) Show that the Fourier coefficients satisfy the inequality $|\hat{c}_n| \leq \|f\|/\|\varphi_n\|$.

$$|\hat{c}_n| = \frac{|\langle f, \varphi_n \rangle|}{\|\varphi_n\|^2} \overset{\text{Cauchy-Schwarz}}{\leq} \frac{\|f\|\,\|\varphi_n\|}{\|\varphi_n\|^2} = \frac{\|f\|}{\|\varphi_n\|}.$$

(c) Show that $\left\| f - \sum_{n=1}^{N} \hat{c}_n \varphi_n \right\|^2 = \|f\|^2 - \sum_{n=1}^{N} |\hat{c}_n|^2 \, \|\varphi_n\|^2$.

$$\left\| f - \sum_{n=1}^{N} \hat{c}_n \varphi_n \right\|^2 = \left\langle f - \sum_{n=1}^{N} \hat{c}_n \varphi_n ,\ f - \sum_{n=1}^{N} \hat{c}_n \varphi_n \right\rangle$$

$$= \langle f, f \rangle - \sum_{n=1}^{N} \hat{c}_n \langle \varphi_n, f \rangle - \sum_{n=1}^{N} \hat{c}_n \langle f, \varphi_n \rangle + \underbrace{\sum_{n=1}^{N} \hat{c}_n^{\,2} \langle \varphi_n, \varphi_n \rangle}_{\text{by orthogonality.}}$$

$$= \|f\|^2 - \underbrace{2 \sum_{n=1}^{N} \hat{c}_n^{\,2} \|\varphi_n\|^2}_{\text{def. of Fourier coef.}} + \sum_{n=1}^{N} \hat{c}_n^{\,2} \|\varphi_n\|^2$$

$$= \|f\|^2 - \sum_{n=1}^{N} \hat{c}_n^{\,2} \|\varphi_n\|^2.$$

Continued...

(d) Write Bessel's inequality satisfied by the Fourier coefficients $\hat{c}_n$ and prove it using part (b).

$$\left\| f - \sum_{n=1}^{N} \hat{c}_n \varphi_n \right\|^2 \geq 0 \implies$$

$$\forall N, \quad \underbrace{\sum_{n=1}^{N} \hat{c}_n^2 \|\varphi_n\|^2}_{} \leq \|f\|^2 \quad - \text{ Bessel's inequality for } (\varphi_n)_{n=1}^{N}.$$

the series converges, since partial sums are monotonically increasing

$$\sum_{n=1}^{\infty} \hat{c}_n^2 \|\varphi_n\|^2 \leq \|f\|^2. \quad - \text{ Bessel's inequality for } (\varphi_n)_{n=1}^{\infty}$$

(e) Write Parseval's equality and show that it implies the mean-square convergence of the partial sum $\sum_{n=1}^{N} \hat{c}_n \varphi_n$.

$$\sum_{n=1}^{\infty} \hat{c}_n^2 \|\varphi_n\|^2 = \|f\|^2. \quad - \text{ Parseval's equality.}$$

$$\iff \sum_{n=1}^{N} \hat{c}_n^2 \|\varphi_n\|^2 \xrightarrow[N \to \infty]{} \|f\|^2$$

$$\iff \left\| f - \sum_{n=1}^{N} \hat{c}_n \varphi_n \right\|^2 \longrightarrow 0$$

(mean-square convergence.)

5. (16 points) (a) Let $f(x)$, $-\pi \le x \le \pi$ be an odd function with $f(\pm\pi) = 0$ and $f''(x)$ continuous for $-\pi \le x \le \pi$. Show that the Fourier coefficients of f satisfy

$$A_n = 0, \quad B_n = -\frac{1}{n^2}\frac{2}{\pi}\int_0^\pi f''(x)\sin nx\,dx.$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}}dx = 0. \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{\text{odd}}\,dx = 0.$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi f(x)\left(-\frac{1}{n}\cos nx\right)'dx$$

$$= \frac{2}{\pi}\Big[\underbrace{f(x)\left(-\frac{1}{n}\cos nx\right)\Big]_{x=0}^{x=\pi}}_{=0,}+\frac{2}{\pi}\frac{1}{n}\int_0^\pi f'(x)\cos nx\,dx$$

$f(\pi)=0$, $f(0)=0$
since f is odd
and coutinuous.

$$= \frac{2}{\pi}\frac{1}{n}\int_0^\pi f'(x)\left(\frac{1}{n}\sin nx\right)'dx$$

$$= \frac{2}{\pi}\frac{1}{n^2}\underbrace{\Big[f'(x)\sin nx\Big]_{x=0}^{x=\pi}}_{=0,} - \frac{2}{\pi}\frac{1}{n^2}\int_0^\pi f''(x)\sin nx\,dx$$

$\sin 0 = 0$, $\sin n\pi = 0$

$$= -\frac{1}{n^2}\frac{2}{\pi}\int_0^\pi f''(x)\sin nx\,dx.$$

Continued...

(b) Let $f(x) = x(\pi - x)$, $0 \le x \le \pi$. Compute the <u>Fourier sine series</u> of f. [Hint: use part (a) to shorten the computation.]

$$B_n = -\frac{1}{n^2}\frac{2}{\pi}\int_0^\pi \overbrace{f''(x)}^{'' -2} \sin nx \, dx$$

$$= +\frac{1}{n^2}\frac{4}{\pi}\int_0^\pi \sin nx \, dx$$

$$= \frac{1}{n^2}\frac{4}{\pi}\left(-\frac{1}{n}\cos nx\right)_{x=0}^{x=\pi}$$

$$= \frac{4}{\pi n^3}\left(1-(-1)^n\right)$$

$$f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin nx$$

$$= \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin(2k-1)x$$

$$= \frac{8}{\pi}\left(\sin x + \frac{1}{27}\sin 3x + \frac{1}{125}\sin 5x + \cdots\right)$$

(c) Differentiate the Fourier sine series in part (b) term-by-term. Find the sum $S(x)$ of the resulting series and sketch a graph of $S(x)$ on the interval $-2\pi \leq x \leq 2\pi$.

$$f'(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$$

$$= \frac{8}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right)$$

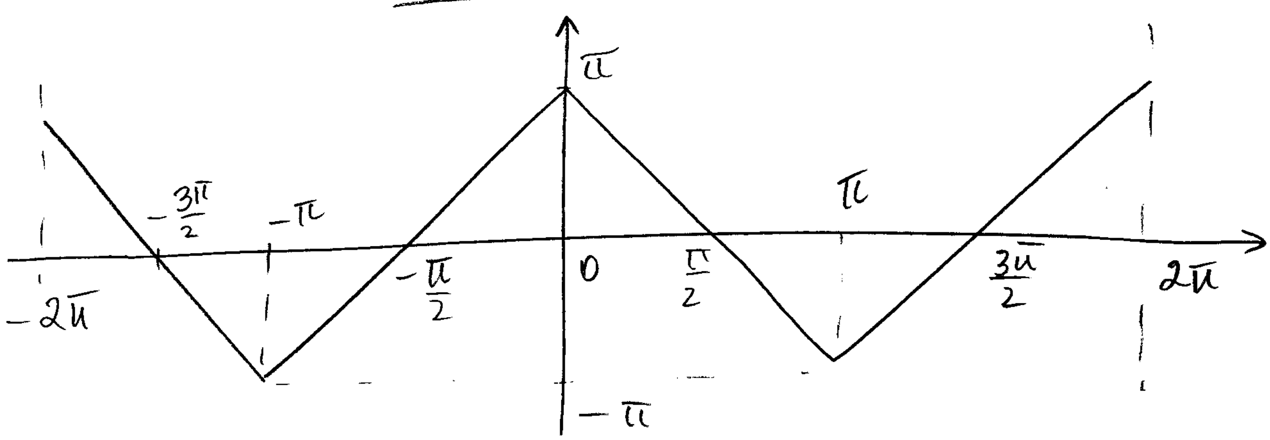
(cosine series)

For $0 < x < \pi$

the series represents $f'(x) = \pi - 2x$,

for x outside this interval

the series converges to the even periodic extension of $f'(x)$.



6. (bonus: 6 points) Let $f(x) = x(\pi - x)$, $0 \le x \le \pi$. Compute the mean square error of the Fourier sine series, $\sigma_N^2 = \frac{1}{\pi}\int_0^\pi \left(f(x) - \sum_{n=1}^N B_n \sin nx\right)^2 dx$ and show that $\sigma_N^2 = O(\frac{1}{N^5})$.

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty} B_n^2 = \frac{1}{2}\sum_{n=N+1}^{\infty} \left(\frac{4}{\pi}\right)^2 \left(\frac{1-(-1)^n}{n^3}\right)^2$$

or

$$\sigma_{2N}^2 = \sigma_{2N-1}^2 = \frac{1}{2}\sum_{k=N+1}^{\infty} \left(\frac{8}{\pi}\right)^2 \frac{1}{(2k-1)^6}$$

$$= \frac{32}{\pi^2}\sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^6} \le \frac{32}{\pi^2}\int_N^\infty \frac{1}{(2x-1)^6}\,dx$$

$$= \frac{32}{\pi^2}\left(-\frac{1}{10}\right)\left[\frac{1}{(2x-1)^5}\right]_{x=N}^{x=\infty}$$

$$= \frac{16}{5\pi^2}\,\frac{1}{(2N-1)^5} = O\left(\frac{1}{N^5}\right)$$

$$\left[\text{since } (2N-1)^5 \ge \left(\frac{N}{2}\right)^5 \Rightarrow \frac{1}{(2N-1)^5} \le \frac{32}{N^5}.\right]$$

The end.