9/27/2012

MATH 480: PARTIAL DIFFERENTIAL EQUATIONS

Convergence of Fourier series

$f: (-L, L) \to \mathbb{R}$ - piecewise smooth

(values at end points or points of discontinuity do not matter, we can consider it arbitrarily defined there.)

$$f_P(x) = f\left(x - 2L\left[\frac{x+L}{2L}\right]\right),$$ where

$$[z] = \sup\{n \in \mathbb{Z} : n \leq z\}.$$

- the periodic extension of f

$(f_P(x) = f(x),\ x \in (-L, L),\ f(x+2L) = f(x),\ x \in \mathbb{R})$

$$S_N(x) = A_0 + \sum_{n=1}^{N} A_n \cos\frac{\pi n x}{L} + B_n \sin\frac{\pi n x}{L}$$ - partial sum

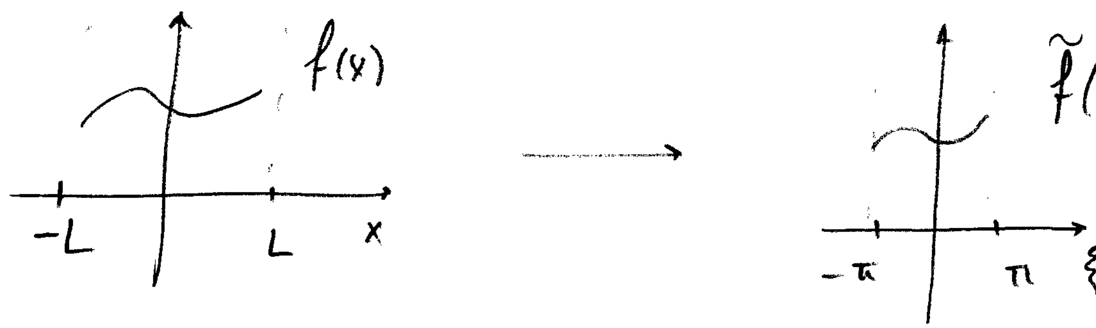
$$S(x) = \lim_{N\to\infty} S_N(x)$$ - sum of the Fourier series.

Theorem (Convergence of Fourier series.)

Let f be piecewise smooth then $\forall x \in \mathbb{R}$ $S(x)$ is defined and

$$S(x) = \frac{1}{2}\left[f_P(x+0) + f_P(x-0)\right].$$

Scaling property.



$$\xi = \frac{\pi x}{L} \; ; \qquad \xi \in (-\pi, \pi)$$

$$f(x) = \tilde f(\xi).$$

If $\quad f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{L} + B_n \sin\frac{\pi n x}{L}$

then $\tilde f(\xi) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\xi + B_n \sin n\xi$

Since

$$\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde f(\xi)\cos n\xi\, d\xi.$$

Thus, it is enough to consider the case $L = \pi$; if the convergence theorem is true for $(-\pi, \pi)$ then it is true for any other value of L by rescaling.

Lemma (Riemann - Lebesgue)

If f is integrable on (a, b) then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0$$

and $$\lim_{\lambda \to \infty} \int_a^b f(x) \cos \lambda x \, dx = 0.$$

Proof: When $f(x)$ is piecewise smooth:

$$\int_a^b f(x) \sin \lambda x \, dx = \int_a^b f(x) \left(-\frac{1}{\lambda} \cos \lambda x\right)' dx$$

$$= \sum_{i=0}^{p} \int_{x_i}^{x_{i+1}} f(x) \left(-\frac{1}{\lambda} \cos \lambda x\right)' dx$$

$$= \sum_{i=0}^{p} \underbrace{\left[-\frac{1}{\lambda} f(x) \cos \lambda x\right]_{x_i}^{x_{i+1}}}_{\to 0,\ \lambda \to \infty} + \underbrace{\frac{1}{\lambda} \underbrace{\int_{x_i}^{x_{i+1}} f'(x) \cos \lambda x \, dx}_{\text{bounded}, \lambda \to \infty}}_{\to 0,\ \lambda \to \infty}$$

When $f(x)$ is integrable, take a suitable lower Darboux sum:

$$0 \le \int_a^b f(x) dx - \sum m_i \Delta x_i < \varepsilon$$

Then define a piecewise constant function

$$f_\varepsilon(x) = m_i, \quad x \in I_i$$

$$\Rightarrow \int_a^b f(x) \sin \lambda x \, dx = \overbrace{\int_a^b (f(x) - f_\varepsilon(x)) \sin \lambda x}^{|\ | < \varepsilon}$$

$$+ \underbrace{\int_a^b f_\varepsilon(x) \sin \lambda x}_{\to 0,\ \lambda \to \infty}$$

<u>Remark</u>: Another proof of the R-L Lemma follows from the Bessel inequality:

$$2LA_0^2 + L\sum_{n=1}^{N} A_n^2 + B_n^2 \leq \int_{-L}^{L} f(x)^2 dx .$$

Therefore the series $\sum_{n=1}^{\infty} A_n^2$, $\sum_{n=1}^{\infty} B_n^2$ are convergent $\Rightarrow A_n, B_n \to 0$

$$\Rightarrow \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}dx , \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}dx \xrightarrow[n\to\infty]{} 0 .$$

Goal: to prove that

$$S_N(x) \longrightarrow \frac{1}{2}\left[f_p(x+0)+f_p(x-0)\right] .$$

Compute: $(L=\pi !)$

$$S_N(x) = A_0 + \sum_{n=1}^{N} A_n \cos\frac{\pi n x}{L} + B_n \sin\frac{\pi n x}{L}$$

$$\frac{1}{L}\int_{-L}^{L} f(t)\left(\frac{1}{2}+\sum_{n=1}^{N}\cos\frac{\pi n t}{L}\cos\frac{\pi n x}{L} + \sin\frac{\pi n t}{L}\sin\frac{\pi n x}{L}\right)dt$$

(unhappy notation; need to use $L=\pi$!)

$$= \frac{1}{L}\int_{-L}^{L} f(t)\left[\frac{1}{2}+\sum_{n=1}^{N}\cos\frac{\pi n}{L}(t-x)\right]dt$$

$\left[\underline{\text{Def}}: D_N(u) = \frac{1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{N}\cos nu\right)\right]$

$$= \frac{\pi}{L}\int_{-L}^{L} f(t)\, D_N\left(\frac{\pi n}{L}(t-x)\right)dt .$$

Lemma

$$\frac{1}{2} + \cos\alpha + \dots + \cos N\alpha = \frac{\sin\left(N+\frac{1}{2}\right)\alpha}{2\sin\frac{\alpha}{2}},$$

$$\alpha \neq 2\pi n, \quad n \in \mathbb{Z}.$$

Proof: Use

$$1 + r + \dots + r^N = \frac{r^{N+1} - 1}{r - 1}$$

with $r = e^{i\alpha} = \cos\alpha + i\sin\alpha$; then

$$r^k = e^{ik\alpha} = \cos k\alpha + i\sin k\alpha$$

$$\frac{1}{2} + \cos\alpha + \dots + \cos N\alpha = -\frac{1}{2} + \operatorname{Re}\left[1 + \dots + e^{iN\alpha}\right]$$

$$= -\frac{1}{2} + \operatorname{Re}\frac{e^{i(N+1)\alpha} - 1}{e^{i\alpha} - 1},$$

$$\frac{e^{i(N+1)\alpha} - 1}{e^{i\alpha} - 1} = \frac{e^{i\alpha/2}\left(e^{i(N+\frac{1}{2})\alpha} - e^{-i\alpha/2}\right)}{e^{i\alpha/2}\left(e^{i\alpha/2} - e^{-i\alpha/2}\right)}$$

$$= \frac{1}{2i\sin\frac{\alpha}{2}}\left(e^{i(N+\frac{1}{2})\alpha} - e^{-i\alpha/2}\right)$$

$$= \frac{1}{2i\sin\frac{\alpha}{2}}\left(\cos\left(N+\tfrac{1}{2}\right)\alpha + i\sin\left(N+\tfrac{1}{2}\right)\alpha - \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$\operatorname{Re}\frac{e^{i(N+1)\alpha} - 1}{e^{i\alpha} - 1} = \frac{1}{2\sin\frac{\alpha}{2}}\left(\sin\left(N+\tfrac{1}{2}\right)\alpha + \sin\frac{\alpha}{2}\right)$$

$$= \frac{\sin\left(N+\frac{1}{2}\right)\alpha}{2\sin\frac{\alpha}{2}} + \frac{1}{2}.$$

Another way: $\qquad \left[\cos a \sin b = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right) \right]$

$$\frac{1}{2}\sin\alpha + \sin\alpha\cos\alpha + \ldots + \sin\alpha\cos N\alpha =$$

$$= \cancel{\frac{1}{2}\sin\alpha} + \frac{1}{2}(\sin 2\alpha - 0) + \frac{1}{2}(\sin 3\alpha - \cancel{\sin\alpha}) + \ldots$$

$$\ldots + \frac{1}{2}\left(\sin(N+1)\alpha - \sin(N-1)\alpha\right)$$

$$= \frac{1}{2}\left(\sin(N+1)\alpha + \sin N\alpha\right) = \sin\left(N+\tfrac{1}{2}\right)\alpha \cos\frac{\alpha}{2}$$

$$\left[a = \frac{\alpha}{2},\ b = \left(N+\tfrac{1}{2}\right)\alpha \right]$$

Third way:

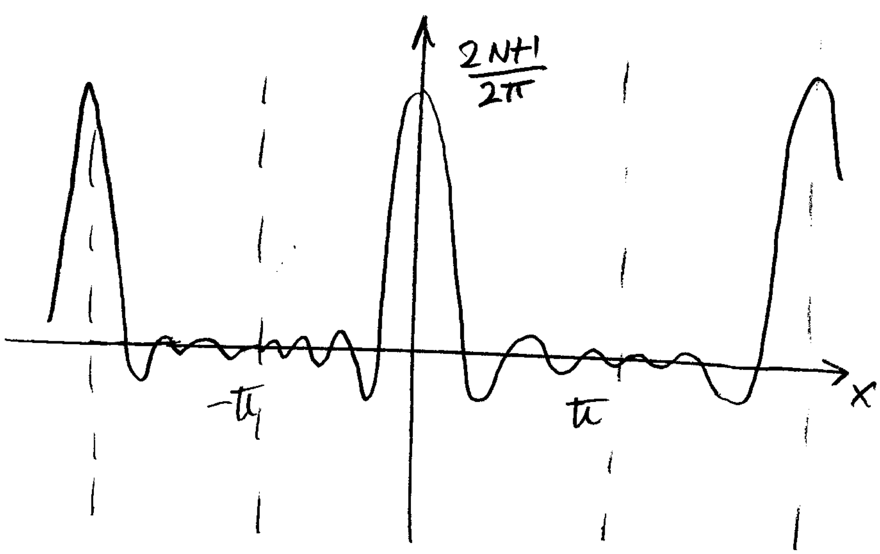
$$2\left(\frac{1}{2} + \cos\alpha + \ldots + \cos N\alpha\right) = e^{-iN\alpha} + e^{-i(N-1)\alpha} + \ldots + e^{i(N-1)\alpha} + e^{iN\alpha}$$

$$= \frac{e^{i(N+1)\alpha} - e^{-iN\alpha}}{e^{i\alpha} - 1} = \frac{e^{i(N+\frac{1}{2})\alpha} - e^{-i(N+\frac{1}{2})\alpha}}{e^{i\alpha/2} - e^{-i\alpha/2}}$$

$$= \frac{\sin\left(N+\frac{1}{2}\right)\alpha}{\sin\frac{\alpha}{2}}.$$

Def: The Dirichlet kernel:

$$D_N(u) = \frac{1}{\pi}\left(\frac{1}{2} + \sum_{N=1}^{N} \cos nu\right) = \begin{cases} \dfrac{1}{2\pi}\dfrac{\sin\left(N+\frac{1}{2}\right)\alpha}{\sin\frac{\alpha}{2}} & \alpha \neq 2\pi n \\[2ex] \dfrac{2N+1}{2\pi}, & \alpha = 2\pi n \end{cases}$$



Note:

$$\int_{-\pi}^{\pi} D_N(u)\,du = 1$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi}\,du$$

$D_N(u)$ is 2π-periodic

(clear from $D_N(u) = \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N} \cos nu\right)$).

Further compute:

$$S_N(x) = \int_{-\pi}^{\pi} f(t)\, D_N(t-x)\, dt = \int_{-\pi+x}^{\pi+x} f(x+u)\, D_N(u)\, du$$

$$= \int_{-\pi}^{\pi} f_p(x+u)\, D_N(u)\, du = \int_{-\pi}^{0} f_p(x+u)\, D_N(u)\, du$$

$$+ \int_0^{\pi} f_p(x+u)\, D_N(u)\, du.$$

Lemma (To finish off the proof of Conver. Thm.)

$$\int_0^{\pi} f_p(x+u)\, D_N(u)\, du \longrightarrow \frac{1}{2} f_p(x+0)$$

$$\int_{-\pi}^{0} f_p(x+u)\, D_N(u)\, du \longrightarrow \frac{1}{2} f_p(x-0)$$

Remark: $D_N(u)$ is a δ-like sequence of periodic functions:

$$\int_{-\pi}^{\pi} D_N(u)\, du = 1;$$

$$\int_{-\pi}^{\pi} D_N(u)\, f(u)\, du \longrightarrow f(0)$$

for any smooth function f.

Proof of lemma, for the first integral. [the other one is similar!]

Note: $\int_0^{\pi} D_N(u)\,du = \int_{-\pi}^{0} D_N(u)\,du = \frac{1}{2}$.

$$\int_0^{\pi} f_P(x+u) D_N(u)\,du - \frac{1}{2} f_P(x+0) =$$

$$= \int_0^{\pi} \big(f_P(x+u) - f_P(x+0)\big) D_N(u)\,du =$$

$$= \int_0^{\pi} \underbrace{\frac{f_P(x+u) - f_P(x+0)}{u}}_{=g(u)} \underbrace{\frac{u}{2\pi \sin\frac{u}{2}}}_{U(u)} \sin\left(N+\tfrac{1}{2}\right)u\,du$$

Claim: $g(u), g'(u), U(u), U'(u)$ are piecewise continuous on $(0;\pi)$.

[Once this is established, the proof of lemma is completed by application of Riemann-Lebesgue lemma.]

$U(u)$ is continuous on $(0,\pi]$,

$$\lim_{u \downarrow 0} U(u) = \frac{1}{\pi} \lim_{u \downarrow 0} \frac{u/2}{\sin u/2} = \frac{1}{\pi}.$$

$$U'(u) = \frac{1}{2\pi \sin\frac{u}{2}} - \frac{\frac{u}{2}\cos\frac{u}{2}}{2\pi\left(\sin\frac{u}{2}\right)^2}$$

$$= \frac{1}{2\pi} \frac{\sin\frac{u}{2} - \frac{u}{2}\cos\frac{u}{2}}{\left(\sin\frac{u}{2}\right)^2} \xrightarrow[u \uparrow 0]{} 0$$

$$g(u) = \frac{f_p(x+u) - f_p(x+0)}{u}$$

$$\lim_{u \downarrow 0} g(u) = \lim_{u \downarrow 0} \frac{f_p'(x+u)}{1} = f_p'(x+0).$$

$$g'(u) = \frac{u f_p'(x+u) - f_p(x+u) + f_p(x+0)}{u^2}$$

$$\lim_{u \downarrow 0} g'(u) = \lim_{u \downarrow 0} \frac{u f_p''(x+u)}{2u} = f_p''(x+0).$$

$\square$