MATH 480 – PARTIAL DIFFERENTIAL EQUATIONS

Orthogonal functions.

($f \perp g$) – notation

Def: f, g are orthogonal if $\langle f, g \rangle = 0$, i.e.

$$\int_0^L f(x)g(x)\,dx = 0.$$

Example: $\sin x$, $\cos x$ on $(0, \pi)$

$$\langle \sin, \cos \rangle = \int_0^\pi \sin x \cos x \, dx = \left(\frac{1}{2}\sin^2 x\right)_0^\pi = 0$$

– orthogonal

$\sin x$, $\cos x$ on $(0, \frac{\pi}{2})$:

$$\langle \sin, \cos \rangle = \int_0^{\pi/2} \sin x \cos x \, dx = \left(\frac{1}{2}\sin^2 x\right)_0^{\pi/2} = \frac{1}{2}.$$

– not orthogonal.

Def: A family of functions (f_i) is orthogonal, if $\langle f_i, f_j \rangle = 0$, $\forall i \neq j$.

A family is orthonormal if it is orthogonal, and $\forall i$ $\|f_i\| = 1$.

Remarks: 1) If (f_i) is orthogonal, and $\forall i$ $\|f_i\| \neq 0$ then (f_i) is <u>linearly independent</u>.

2) If (f_i) is orthogonal, $\forall i$ $\|f_i\| \neq 0$ then (g_i) : $g_i = \frac{f_i}{\|f_i\|}$ is orthonormal.

Exercise: Give a proof of 1) and 2) above.

<u>Examples</u>: 1) $1, \sin x, \cos x$ is an orthogonal family on $(0,\pi)$.

It is not orthonormal;

$$\|1\|^2 = \pi, \quad \|\sin\|^2 = \int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}$$

$$\|\cos\|^2 = \int_0^\pi \cos^2 x\,dx = \frac{\pi}{2}.$$

The family $\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\cos x$

is orthonormal on $(0,\pi)$.

2) Let $\varphi_n(x) = \sin\frac{\pi n x}{L}$, $x \in (0, L)$, $n = 1, 2 \dots$

Then $\varphi_n(x)$ is an infinite orthogonal family of functions.

Indeed,

$$\langle \varphi_n, \varphi_m \rangle = \int_0^L \sin\frac{\pi n x}{L} \sin\frac{\pi m x}{L}\,dx$$

Note: $\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$

$\Big[$ since $\cos(a-b) = \cos a \cos b + \sin a \sin b$
$\cos(a+b) = \cos a \cos b - \sin a \sin b$ $\Big]$

$$\langle \varphi_n, \varphi_m \rangle = \frac{L}{\pi}\int_0^\pi \sin ny \sin my\,dy$$

$$= \frac{L}{2\pi}\int_0^\pi [\cos(n-m)y - \cos(n+m)y]\,dy$$

$$= \begin{cases} \frac{L}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

$\Rightarrow$ (φ_n) orthogonal family, and also

$$\|\varphi_n\| = \sqrt{\frac{L}{2}}, \quad n = 1, 2 \ldots$$

A basis in a finite-dim. vector space V is a set of vectors which is linearly indep. and complete (or generating)

In the infinite-dimensional vector space of functions on $(0, L)$, the set $\{\varphi_n\}$ is linearly independent. It is also complete in the sense to be made precise later.

So far, we will look at Fourier series as infinite linear combinations of trigonometric functions, and worry about convergence later.

Example: $u_0(x) = \frac{x}{L}\left(1 - \frac{x}{L}\right)$; write

$$u_0(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x).$$

If such representation is at all possible, then

$$\langle u_0, \varphi_i \rangle = \overbrace{a_1 \langle \varphi_1, \varphi_i \rangle}^{=0} + \overbrace{a_2 \langle \varphi_2, \varphi_i \rangle}^{=0} + \ldots$$

$$\ldots + a_i \langle \varphi_i, \varphi_i \rangle + \underbrace{\ldots}_{=0}$$

Then $a_i = \dfrac{\langle u_0, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle}$ — Fourier coefficient.

Compute:

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$$a_n = \frac{\langle u_0, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{2}{L}\int_0^L \frac{x}{L}\left(1-\frac{x}{L}\right)\sin\frac{\pi n x}{L}\,dx$$

$$= \frac{2}{L}\,\frac{L}{\pi}\,\frac{1}{\pi^2}\int_0^\pi y(\pi-y)\sin ny\,dy.$$

$$\int_0^\pi y(\pi-y)\sin ny\,dy = \int_0^\pi y(\pi-y)\left(-\frac{1}{n}\cos ny\right)'dy$$

$$= \underbrace{\left[y(\pi-y)\left(-\frac{1}{n}\cos ny\right)\right]_0^\pi}_{=0} + \frac{1}{n}\int_0^\pi(\pi-2y)\cos ny\,dy$$

$$= \frac{1}{n^2}\int_0^\pi(\pi-2y)(\sin ny)'dy =$$

$$= \underbrace{\frac{1}{n^2}\left[(\pi-2y)\sin ny\right]_0^\pi}_{=0} + \frac{1}{n^2}\int_0^\pi 2\sin ny\,dy$$

$$= \frac{2}{n^3}\left[-\cos ny\right]_0^\pi = \frac{2}{n^3}(1-\cos\pi n) = \frac{2}{n^3}(1-(-1)^n).$$

Then $a_n = \dfrac{4}{(\pi n)^3}(1-(-1)^n)$.

and

$$u_0(x) = \frac{x}{L}\left(1-\frac{x}{L}\right) = \sum_{n=1}^\infty \frac{4(1-(-1)^n)}{(\pi n)^3}\sin\frac{\pi n x}{L}$$

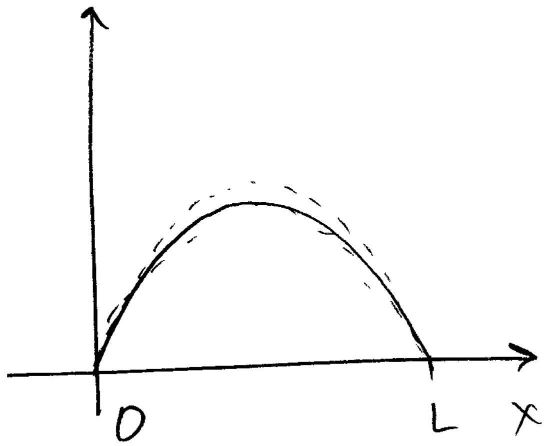
By linear superposition, the solution of the heat equation with $u|_{t=0} = u_0(x)$ is then

$$u(x,t) = \sum_{n=1}^\infty \frac{4(1-(-1)^n)}{(\pi n)^3}\sin\frac{\pi n x}{L}\,e^{-\left(\frac{\pi n}{L}\right)^2 t}.$$

Illustrate:

$$\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{(\pi n)^3} \sin\frac{\pi n x}{L} = \underset{(n=1)}{\frac{8}{\pi^3}\sin\frac{\pi x}{L}} + \underset{(n=3)}{\frac{8}{\pi^3\cdot 9}\sin\frac{3\pi x}{L}}$$

$$+ \underset{(n=5)}{\frac{8}{\pi^3\cdot 125}\sin\frac{5\pi x}{L}} + \underset{(n=7)}{\frac{8}{\pi^3\cdot 343}\sin\frac{7\pi x}{L}} + \dots$$



Consecutive terms in the Fourier series provide improved approximation of $u_0(x)$ over the interval $0<x<L$.

Another approach to definition of Fourier coefficients:

Fix N; consider the problem of finding "best possible" approximation of a function $f(x)$ by a trig polynomial:

$$f(x) \approx c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_N\varphi_N(x)$$

where $\varphi_n(x) = \sin\frac{\pi n x}{L}$, $n = 1, 2 \dots$

<u>Method of least squares</u>:

Determine $c_1, c_2 \dots c_N$ so that

$$\| f - (c_1\varphi_1 + \dots + c_N\varphi_N) \|^2$$

is as small as possible.

More generally, let (φ_n) be an orthogonal set of functions, $\varphi_n \neq 0$ (none of them identically zero)

Define:

$$D(c_1 \dots c_N) = \| f - (c_1\varphi_1 + \dots + c_N\varphi_N)\|^2$$

and look for $c_1 \dots c_N$ that provides the minimum.

Proposition: 1) The minimum of $D(c_1 \dots c_N)$ is attained uniquely when

$$c_n = \frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} \qquad \text{(the Fourier coefficients)}$$

2) The minimum distance is given by

$$d^2_{min}(f) = \|f\|^2 - \sum_{n=1}^{N} \frac{\langle \varphi_n, f\rangle^2}{\langle \varphi_n, \varphi_n\rangle}$$

$$= \|f\|^2 - \sum_{n=1}^{N} c_n^2 \|\varphi_n\|^2$$

3) The Fourier coefficients satisfy Bessel's inequality

$$\sum_{n=1}^{N} c_n^2 \|\varphi_n\|^2 \leq \|f\|^2.$$

Proof: Compute:

$$D(c_1 \dots c_N) = \|f\|^2 - 2\sum_{n=1}^{N} c_n \langle f, \varphi_n\rangle + \sum_{n=1}^{N} c_n^2 \|\varphi_n\|^2$$

[complete the square:]

$$= \|f\|^2 + \sum_{n=1}^{N} \|\varphi_n\|^2 \left(c_n^2 - 2c_n \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2} + \frac{\langle f, \varphi_n\rangle^2}{\|\varphi_n\|^4}\right) - \sum_{n=1}^{N} \frac{\langle f, \varphi_n\rangle^2}{\|\varphi_n\|^4}.$$

$$= \sum_{n=1}^{N} \|\varphi_n\|^2 \left(c_n - \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2}\right)^2 + \|f\|^2 - \sum_{n=1}^{N} \frac{\langle f, \varphi_n\rangle^2}{\|\varphi_n\|^2}.$$

Thus, the min. is achieved when

$$c_n = \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2}.$$

The value of the minimum is

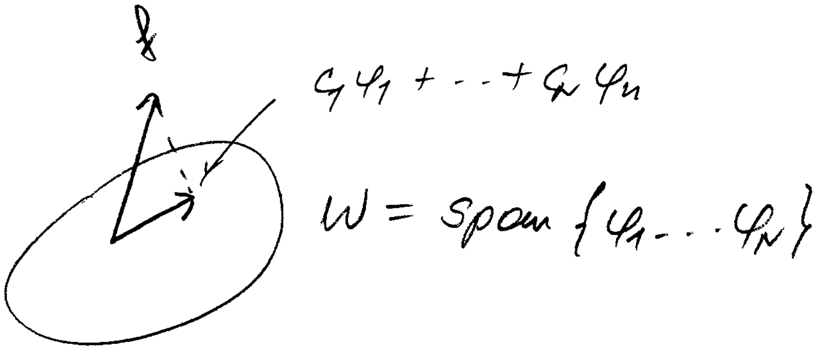
$$d^2_{min}(f) = \|f\|^2 - \sum_{n=1}^{N} \frac{\langle f, \varphi_n\rangle^2}{\|\varphi_n\|^2}$$

$$= \|f\|^2 - \sum_{n=1}^{N} c_n^2 \|\varphi_n\|^2 \geq 0.$$

<u>Remark</u>: If $c_1 \dots c_N$ are Fourier coefficients of f, then

$$f(x) - (c_1\varphi_1(x) + \dots + c_N\varphi_N(x))$$ is orthogonal to all of $\varphi_1 \dots \varphi_N$.

Illustration:



<u>Example</u>: $(1, \cos x, \sin x)$ – orthogonal set of functions on $(-\pi, \pi)$.

Find the optimal least squares approximation of $f(x) = x$!

$$x = c_1 + c_2 \cos x + c_3 \sin x$$

$$c_1 = \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\,dx = 0.$$

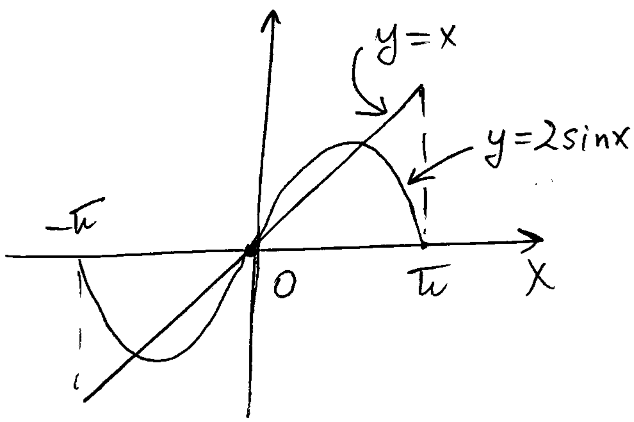
$$c_2 = \frac{\langle x, \cos x \rangle}{\langle \cos x, \cos x \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x\,dx = 0$$

$$c_3 = \frac{\langle x, \sin x \rangle}{\langle \sin x, \sin x \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \,(-\cos x)'\,dx$$

$$= \frac{1}{\pi} \left[-x \cos x\right]_{-\pi}^{\pi} + \frac{1}{\pi} \underbrace{\int_{-\pi}^{\pi} \cos x\,dx}_{=0}$$

$$= \frac{1}{\pi}(\pi + \pi) = 2$$



Thus, the orthogonal projection of $f(x) = x$ onto $W = \text{span}(1, \cos x, \sin x)$ is the function

$$g(x) = 2\sin x.$$