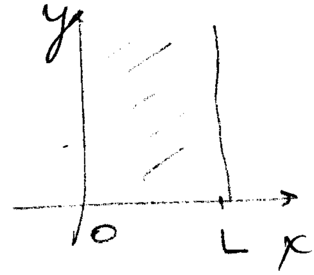
9/6/2012 - 9/11/2012

MATH 480 - PARTIAL DIFFERENTIAL EQUATIONS (1)

Separated solutions with boundary cond

Example: Laplace's equation

$u_{xx} + u_{yy} = 0$



$(x,y) \in (0,L) \times (0,\infty)$

boundary cond:

$$u(0,y) = u(L,y) = 0$$

$$u(x,0) = 0$$

$$u(x,y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky) \\ (A_1 x + A_2)(A_3 y + A_4) \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \cosh ly + A_4 \sinh ly) \end{cases}$$

Case 1

$x = 0 \Rightarrow \quad A_2 (A_3 \cos ky + A_4 \sin ky) = 0$

$A_2 = 0$ or $\underbrace{A_3 = A_4 = 0}_{\text{trivial;}}$.

$x = L \Rightarrow \quad A_1 \sinh kL + A_2 \cosh kL = 0$

$\Rightarrow A_1 = 0$

Case 2

$x = 0 \Rightarrow \quad A_2 (A_3 y + A_4) = 0$

$A_2 = 0$

$x = L \Rightarrow A_1 = 0$.

Case 3

$$x=0 \Rightarrow A_1 (A_3 \cosh ly + A_4 \sinh ly) = 0$$
$$\Rightarrow A_1 = 0$$

$$x=L \Rightarrow A_2 \sin lL \, (A_3 \sinh ly + A_4 \cosh ly)$$

($A_2 = 0 \Rightarrow$ zero solution)

$$\Rightarrow \sin lL = 0$$
$$\Rightarrow lL = \pi n$$
$$l = \frac{\pi n}{L}$$

$$y=0 \Rightarrow A_4 = 0$$

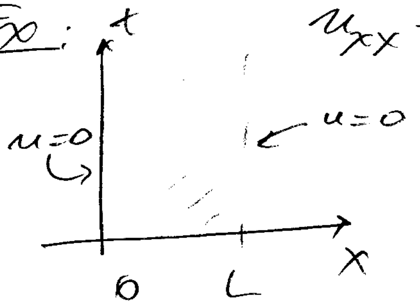
$$A = A_2 A_3 \Rightarrow$$

$$u(x,y) = A \sin \frac{\pi n x}{L} \sinh \frac{\pi n y}{L}$$

heat equation

Ex:

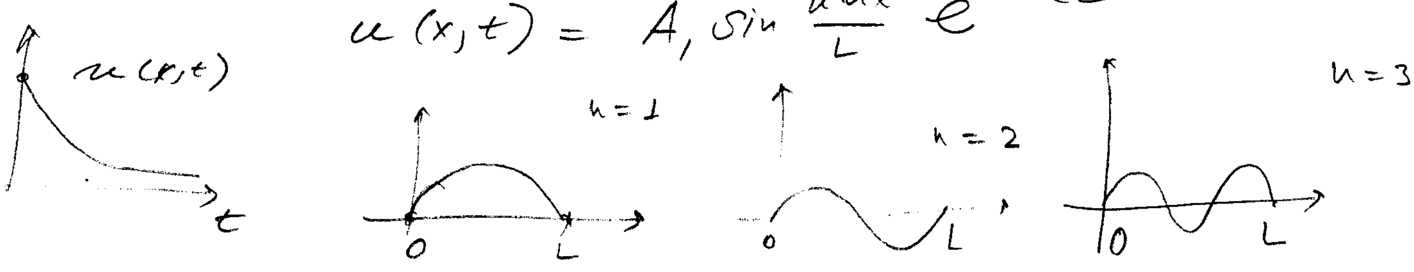
$$u_{xx} - u_t = 0; \quad u(0,t) = 0$$
$$u(L,t) = 0$$



$$u(x,t) = (A_1 \sin \mu x + A_2 \cos \mu x) e^{-\mu^2 t}$$

$$x=0 \Rightarrow A_2 = 0$$
$$x=L \Rightarrow \sin \mu L = 0; \quad \mu = \frac{\pi n}{L}$$

$$u(x,t) = A_1 \sin \frac{\pi n x}{L} e^{-\left(\frac{\pi n}{L}\right)^2 t}$$



<u>Example</u>: $u_{tt} - c^2 u_{xx} = 0$, $(x,t) \in \mathbb{R}^2$

Find complex separated solutions bounded for all t:

$$|u(x,t)| \le M$$

$$u(x,t) = e^{ax+bt};$$

$$b^2 - c^2a^2 = 0 \Rightarrow b = \pm ca$$

$$u(x,t) = e^{ax}e^{cat}, \; e^{ax}e^{-cat}$$

bounded $\Rightarrow a = ik$

$$u(x,t) = e^{ik(x+ct)}, \; e^{ik(x-ct)}$$

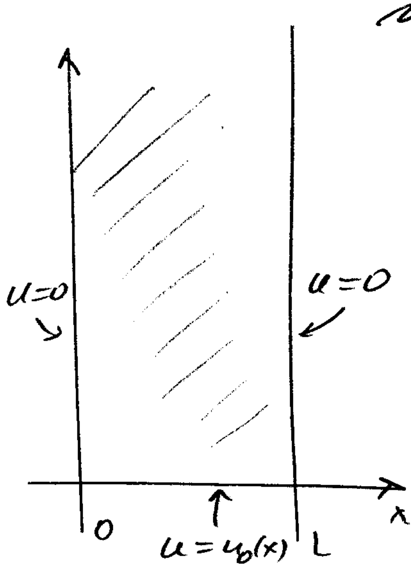
Real solutions:

$$u(x,t) = \begin{cases} \cos k(x+ct) \\ \sin k(x+ct) \\ \cos k(x-ct) \\ \sin k(x-ct) \end{cases}.$$

Back to the heat equation example:

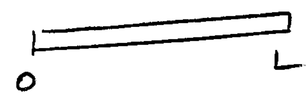
found separated solutions:

$$u_n(x,t) = A_n \sin \frac{\pi n x}{L}\, e^{-\left(\frac{\pi n}{L}\right)^2 t}$$



$$u_{xx} - u_t = 0$$
$$u\big|_{x=0} = u\big|_{x=L} = 0$$
$$u\big|_{t=0} = u_0(x).$$

Problem:



1D heat conducting rod, temp. fixed at the ends;

$u_0(x)$ – initial temperature at each x.

Initial data for u_n:

$$u_n(x,0) = A_n \sin \frac{\pi n x}{L}$$

By linear superposition principle (the PDE is linear, homogeneous!)

if $$u_0(x) = \sum_{n=1}^{N} a_n \sin \frac{\pi n x}{L}$$

for some $a_1 \dots a_N$, then

$$u(x,t) = \sum_{n=1}^{N} a_n \sin \frac{\pi n x}{L}\, e^{-\left(\frac{\pi n}{L}\right)^2 t}$$

is a solution.

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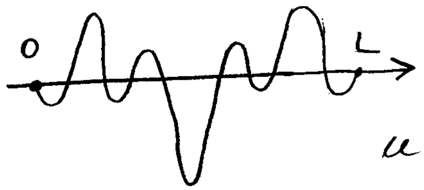
Example: Find a solution of the heat equation $u_{xx} - u_t = 0$



with $u|_{x=0} = u|_{x=L} = 0$,

$$u(x,0) = 5\sin\frac{3\pi x}{L} - 7\sin\frac{9\pi x}{L}$$

Answer:

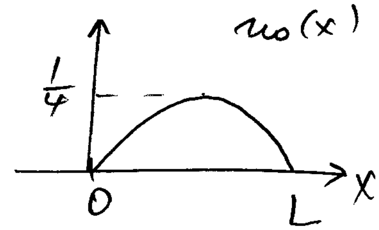
$$u(x,t) = 5\sin\frac{3\pi x}{L}e^{-\left(\frac{3\pi}{L}\right)^2 t} - 7\sin\frac{9\pi x}{L}e^{-\left(\frac{9\pi}{L}\right)^2 t}. \quad \square$$

Linear comb. of trig. functions is a rather restrictive class. How about something more general?

Example: Same problem,

$$u(x,0) = \frac{x}{L}\left(1-\frac{x}{L}\right)$$

(quadratic function.)



Is there a way to approximate

$$u(x,0) \approx \sum_{n=1}^{N} a_n \sin\frac{\pi n x}{L},$$

for certain $a_1 \ldots a_N$?

Answer: yes, there is.

Use <u>Fourier series</u>.

Def: • A Fourier series on an interval $-L<x<L$ is a formal expression:

$$a_0 + \sum_{n=1}^{\infty} a_n \sin\frac{\pi n x}{L} + \sum_{n=1}^{\infty} b_n \cos\frac{\pi n x}{L}.$$

- A Fourier sine series on $0<x<L$ is a formal expression

$$\sum_{n=1}^{\infty} a_n \sin\frac{\pi n x}{L}$$

<u>Remark</u>: The sine series seems like a logical choice for a problem with zero boundary conditions: $u|_{x=0} = u|_{x=L} = 0$ since each term in the series satisfies such boundary condition.

Historically, the heat conduction problem was the motivation for Fourier to introduce this type of series.

<u>Linear algebra</u> V – real vector space; finite-dim. $u_1 \dots u_N$ – basis.

$$v = a_1 u_1 + \dots + a_N u_N$$

$a_1 \dots a_N$ – coordinates.

In finite-dim. case, for arbitrary basis we need to solve a system of linear equations to find coordinates $a_1 \dots a_N$.

Simpler to determine coordinates if a basis is orthogonal; then

$$a_i = \frac{v \cdot u_i}{\|u_i\|^2}.$$

Indeed,

$$v \cdot u_i = \underbrace{a_1 u_1}_{0} + \ldots + a_i u_i \cdot u_i + \ldots + \underbrace{a_N u_N \cdot u_i}_{0}$$

$$= a_i \, u_i \cdot u_i = a_i \|u_i\|^2$$

$$\Rightarrow a_i = \frac{v \cdot u_i}{\|u_i\|^2}.$$

Consider a vector space of functions:

$$V = \{ w : (0, L) \to \mathbb{R} \ ; \ w \text{ is smooth} \}$$

Smooth = differentiable sufficiently many times

(will later extend this to piecewise smooth, and even more discontinuous functions.)

Certainly, $\varphi_n(x) = \sin \frac{\pi n x}{L} \in V$

$n = 1, 2 \ldots$

The idea of Fourier series is that the infinite family $\{\varphi_n(x)\}$ can be viewed as an infinite orthogonal basis in the space V.

Def: The inner product of two real-valued functions on $(0, L)$ is

$$\langle f, g \rangle = \int_0^L f(x) g(x) \, dx$$

<u>Exercise</u>: Check the following properties of the inner product:

symmetry $\langle f, g \rangle = \langle g, f \rangle$

linearity: $\langle c_1 f_1 + c_2 f_2, g \rangle = c_1 \langle f_1, g \rangle + c_2 \langle f_2, g \rangle$

$\langle f, c_1 g_1 + c_2 g_2 \rangle = c_1 \langle f, g_1 \rangle + c_2 \langle f, g_2 \rangle$.

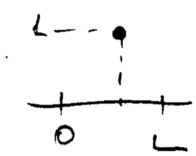
positive-definiteness: $\langle f, f \rangle \geq 0$;

$\langle f, f \rangle = 0 \iff f = 0$

<u>Remark</u>: The last statement is true for smooth (continuous) functions.

More generally, $\int_0^L f(x)^2 \, dx = 0$ implies that $f(x) = 0$ except on a set of <u>zero measure</u>: by definition it is a set which can be covered by a family of intervals of arbitrarily small total length.

<u>Ex</u>:



has integral $= 0$.

<u>Def</u>. The norm $\|f\|$ (or the L^2 norm) of a function f on $(0, L)$, is the nonnegative value defined by

$$\|f\|^2 = \int_0^L f(x)^2 \, dx = \langle f, f \rangle$$

Proposition (The Cauchy-Schwarz inequality.)

$$|\langle f,g\rangle| \le \|f\|\,\|g\|.$$

Proof:

$$0 \le \|f+tg\|^2 = \langle f+tg, f+tg\rangle$$
$$= \langle f,f\rangle + t\langle g,f\rangle + t\langle f,g\rangle + t^2\langle g,g\rangle$$
$$= \langle f,f\rangle + 2t\langle f,g\rangle + t^2\langle g,g\rangle.$$

quadratic in t, min. is achieved when

$$2\langle f,g\rangle + 2t\langle g,g\rangle = 0$$
$$\Rightarrow t = -\frac{\langle f,g\rangle}{\langle g,g\rangle}$$
$$\Rightarrow \quad \langle f,f\rangle - 2\frac{\langle f,g\rangle^2}{\langle g,g\rangle} + \frac{\langle f,g\rangle^2}{\langle g,g\rangle} \ge 0$$
$$\langle f,f\rangle \ge \frac{\langle f,g\rangle^2}{\langle g,g\rangle}$$
$$\Rightarrow \langle f,g\rangle^2 \le \langle f,f\rangle\langle g,g\rangle$$
$$= \|f\|^2\,\|g\|^2. \quad \square$$