9/4/2012

MATH 480 – PARTIAL DIFFERENTIAL EQUATIONS

Separation of variables.

2 variables; 2nd order.

Ex. $u_{xx} + u_{yy} = 0$

$u(x,y) = X(x)\,Y(y)$ – separated solution.

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

both terms are constants, adding up to zero.

$$\frac{X''(x)}{X(x)} = \lambda \; ; \quad \frac{Y''(y)}{Y(y)} = -\lambda \; ;$$

λ – separating const.

$$X''(x) - \lambda X(x) = 0$$
$$Y''(y) - \lambda\, Y(y) = 0$$

Case 1 $\quad \lambda > 0 \Rightarrow \lambda = k^2, \; k > 0$

$$X(x) = A_1 e^{kx} + A_2 e^{-kx}$$
$$Y(y) = A_3 \cos ky + A_4 \sin ky$$

Case 2 $\quad \lambda = 0 \Rightarrow$

$$X(x) = A_1 x + A_2$$
$$Y(y) = A_3 y + A_4.$$

Case 3: $\lambda < 0$ then $\lambda = -l^2$, $l > 0$.

$$X(x) = A_1 \cos lx + A_2 \sin lx$$
$$Y(y) = A_3 e^{ly} + A_4 e^{-ly}.$$

Convenient to introduce hyperbolic functions:

$$\sinh a = \tfrac{1}{2}\left(e^{a} - e^{-a}\right)$$
$$\cosh a = \tfrac{1}{2}\left(e^{a} + e^{-a}\right)$$

$$e^{a} = \cosh a + \sinh a, \quad e^{-a} = \cosh a - \sinh a$$

Then

$$u(x,y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky), & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) & \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \cosh ly + A_4 \sinh ly) & l > 0. \end{cases}$$

– solutions in separated form.

Book, (0.2.3)

<u>Real and complex separated solutions</u>

<u>Proposition</u>: Let $u = v + iw$ – solution of
$Lu = f$; $Lu = a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + h u$,
where a, b, c, d, e, h, f – real functions of x, y.

Then v satisfies $Lu = f$
and w satisfies $Lw = 0$.

Proof: Linearity of derivatives:

$$u_{xx} = v_{xx} + i w_{xx}, \text{ etc}$$

$\Rightarrow$ substitute u in the equation, separate real and imaginary parts.

Ex:

$$u_{xx} + u_{yy} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = 2ik^2$$

$$X'' - 2ik^2 X = 0; \quad Y'' + 2ik^2 Y = 0$$

$$(k(1+i))^2 = 2ik^2 \quad (k(1-i))^2 = -2ik^2$$

$$X(x) = A_1 e^{k(1+i)x} + A_2 e^{-k(1+i)x}$$

$$Y(y) = A_3 e^{k(1-i)y} + A_4 e^{-k(1-i)y}$$

$$u(x,y) = \begin{cases} e^{k(x+y)} e^{ik(x-y)} \\ e^{k(x-y)} e^{ik(x+y)} \\ e^{-k(x-y)} e^{-ik(x+y)} \\ e^{-k(x+y)} e^{-ik(x-y)} \end{cases}$$

Real solutions by taking Re and Im:

$$u(x,y) = \begin{cases} e^{k(x+y)} \cos(k(x-y)), \; e^{k(x+y)} \sin(k(x-y)) \\ e^{k(x-y)} \cos k(x+y), \; e^{k(x-y)} \sin k(x+y) \\ e^{-k(x-y)} \cos k(x+y), \; e^{-k(x-y)} \sin k(x+y) \\ e^{-k(x+y)} \cos k(x-y) \; e^{-k(x+y)} \sin k(x-y). \end{cases}$$

<u>Proposition</u> Consider the linear homogeneous PDE:

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + h u = 0$$

a, b, c, d, e, h – real const.

Then there exist complex separated solutions of the form

$$u(x,y) = e^{\alpha x} e^{\beta y}$$

for appropriate choices of complex numbers α, β.

<u>Proof</u>: Substitute u $\Rightarrow$

$$\left(a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + h\right) e^{\alpha x} e^{\beta y} = 0$$

$$\Rightarrow \quad a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + h = 0$$

For any given β, quadratic eqn. for α
$\Rightarrow$ 2 complex roots.

Likewise, for any given α, 2 cplx solution β.

<u>Examples</u> 1) Find solutions to $u_{xx} - u_t = 0$ in the form $u(x,t) = e^{i\mu x} e^{\beta t}$

$$-\mu^2 - \beta = 0 \; ; \quad \beta = -\mu^2$$

$$u(x,t) = e^{i\mu x} e^{-\mu^2 t}$$

By taking Re and Im, obtain

$$u(x,t) = \sin \mu x \, e^{-\mu^2 t}, \quad \cos \mu x \, e^{-\mu^2 t}$$

general form

$$u(x,t) = A_1 \cos \mu x \, e^{-\mu^2 t} + A_2 \sin \mu x \, e^{-\mu^2 t}.$$

(decay in t, osc. in x)

2) Find solutions of $u_{xx} - u_t = 0$ on the form

$$u(x,t) = e^{\alpha x} e^{i\omega t}, \quad \omega \text{ - real, } > 0.$$

$$\alpha^2 - i\omega = 0 \Rightarrow \quad \alpha = \pm \frac{1+i}{\sqrt{2}} \sqrt{\omega}$$

$$\omega = 2k^2 \Rightarrow \alpha = \pm (1+i) k$$

$$u(x,t) = e^{(1+i)kx} e^{2ik^2 t}, \quad e^{-(1+i)kx} e^{2ik^2 t}$$

$$e^{kx + i(kx + 2k^2 t)}, \quad e^{-kx + i(-kx + 2k^2 t)}$$

$$u(x,t) = e^{\pm kx} \cos(kx + 2k^2 t), \quad e^{\pm kx} \sin(kx + 2k^2 t)$$

Not separated, "quasi-separated"
(real-imaginary parts of a complex separated solution.)

3) $u_x + (x+y) u_y = 0$

$$u(x,y) = X(x) Y(y) \Rightarrow X'Y + (x+y) XY' = 0$$

$$\Rightarrow \quad \frac{X'(x)}{X(x)} + (x+y) \frac{Y'(y)}{Y(y)} = 0.$$

Freeze y $\Rightarrow$ $\dfrac{X'(x)}{X(x)} = cx + d$

Freeze x $\Rightarrow$ $-a + (b+y)\dfrac{Y'(y)}{Y(y)} = 0$

$$\frac{Y'(y)}{Y(y)} = \frac{a}{b+y} \quad \ldots$$

$$\underbrace{\frac{X'(x)}{X(x)}}_{-a} + x\,\frac{Y'(y)}{Y(y)} + y\,\frac{Y'(y)}{Y(y)} = 0$$

$$\Rightarrow \frac{Y'(y)}{Y(y)} = 0 \Rightarrow Y(y) = \text{const.}$$

$$\Rightarrow \frac{X'(x)}{X(x)} = 0 \Rightarrow X(c) = \text{const}$$

No separated solutions except for constants.

4) $u_{xx} + y^2 u_{yy} + y u_y = 0, \quad y > 0.$

$$X''(x)Y(y) + y^2 X(x) Y''(y) + y\, X(x) Y'(y) = 0$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{-\lambda} + y^2\frac{Y''(y)}{Y(y)} + y\,\frac{Y'(y)}{Y(y)} = 0$$

$$\begin{cases} X'' + \lambda X = 0 \\ y^2 Y''(y) + y\,Y'(y) - \lambda Y(y) = 0 \end{cases}$$

<u>Case 1:</u> $\lambda > 0$ $\Rightarrow$ $X = A_1 \cos kx + A_2 \sin kx$
$\lambda = k^2$

$Y(y) = y^r,$

$r(r-1) + r - k^2 = 0$

$r^2 - k^2 = 0 \Rightarrow r = \pm k$

$Y(y) = A_3 y^k + A_4 y^{-k}.$

Case 2: $\lambda = 0 \Rightarrow X = A_1 + A_2 x$

$y^2 Y''(y) + y\, Y'(y) = 0$

$r^2 = 0 \Rightarrow r = 0$

$Y(y) = A_3 + A_4 \log y$

Case 3: $\lambda < 0, \ \lambda = -k^2 \Rightarrow X = A_1 e^{kx} + A_2 e^{-kx}$
$(k > 0)$

$r^2 + k^2 = 0 \Rightarrow r = \pm i k$

$\Rightarrow Y(y) = A_3 \cos(k \log y) + A_4 \sin(k \log y)$