MATH 480 – PARTIAL DIFFERENTIAL EQUATIONS.

Linearity and superposition principle.

Def: $F(x, u, \partial_{x_i} u, \partial_{x_i x_j} u, \dots) = 0$

is linear if F is linear in all arguments except possibly x.

2 vars, 2nd order (to simplify notation)

$$F = a_0(x,y)u + a_1(x,y)u_x + a_2(x,y)u_y + a_{11}(x,y)u_{xx} + a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} + b(x,y) = 0.$$

Introduce formal notation

$$L = a_0 + a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_{11} \frac{\partial^2}{\partial x^2} + a_{12} \frac{\partial^2}{\partial x \partial y} + a_{22} \frac{\partial^2}{\partial y^2}.$$

$$(a_0 = a_0(x,y), \text{ etc.})$$

– linear operator.

Then, a linear PDE takes the form

$$Lu = f \quad ; \quad f(x,y) = -b(x,y).$$

A linear PDE is homogeneous, if $f = 0$.

Ex: $u_x + u u_y = 0$ – not linear (nonlinear)

$u_{xy} = f$ – linear, nonhomogen.

$u_x - u_y = 0$ — lin. transp.

$u_{xx} - u_{yy} = 0$ — wave

$u_{xx} + u_{yy} = 0$ — Laplace

$\Big\}$ linear homogeneous

$u_{xx} + u_{yy} = f \neq 0$ — Poisson's eq.
linear, nonhomogen.

Superposition principle

Thm 1 If $u_1, u_2 \ldots u_N$ - solutions of
$Lu = 0$ (homogeneity is important!)
then $c_1 u_1 + c_2 u_2 + \ldots + c_N u_N$ - solution.

Proof: Verify: $L(u_1 + u_2) = L(u_1) + L(u_2)$
$L(cu) = cL(u)$.
(L is a linear transformation.)

Then $L(c_1 u_1 + \ldots + c_N u_N) = c_1 L(u_1) + \ldots + c_N L(u_N) = 0$

Thm 2. If $LV = 0$, $LW = f$, $u = V + W$
then $Lu = f$

Proof: easy

Thm 3. Suppose u_p is a particular solution of $Lu_p = f$. Then any other solution $Lu = f$ has the form $u = u_p + V$ where V is some solution of $LV = 0$.

Proof: if $Lu_p = f$, $Lu = f$ then $v = u - u_p$ satisfies $Lv = 0$.

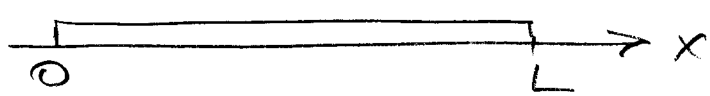
Example: $u_{xx} + u_{yy} = 1$; (Poisson)

$u_p = \frac{x^2}{4} + \frac{y^2}{4}$ is a particular solution.

then *any* solution of (Poisson) is given by $u_p + V$, where

$V_{xx} + V_{yy} = 0$ (Laplace)

The one-dimensional heat equation

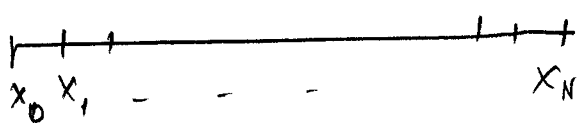


1D rod

$u(x,t)$ – temperature

Equation for $u(x,t)$?

To begin with, consider



($N+1$ points, N equally spaced intervals.)

Reasonably,

$\frac{\partial u}{\partial t}(x_i,t) = 0$ if $u(x_i,t) = \frac{1}{2}\left(u(x_{i-1},t) + u(x_{i+1},t)\right)$

(temperature does not change if it is equal to the average of left and right values.)

Reasonable ODE model:

$$\frac{\partial u}{\partial t}(x_i,t) = k_N\left(\tfrac{1}{2}\left(u(x_{i-1},t)+u(x_{i-1},t)\right) - u(x_i,t)\right)$$

Taylor-expand:

$$u(x_{i+1},t) = u(x_i,t) + (x_{i+1}-x_i)\,u_x(x_i,t) + \tfrac{1}{2}(x_{i+1}-x_i)^2 u_{xx}(x_i',t)$$

$$[\, x_i < x_i' < x_{i+1} \,]$$

$$u(x_{i-1},t) = u(x_i,t) + (x_{i-1}-x_i)\,u_x(x_i,t) + \tfrac{1}{2}(x_{i-1}-x_i)^2 u_{xx}(x_i'',t)$$

$$[\, y_{i-1} < x_i'' < x_i \,]$$

$$x_{i+1}-x_i = \Delta x = \frac{L}{N}\,; \quad x_{i-1}-x_i = -\Delta x$$

Then $\frac{\partial u}{\partial t}(x_i,t) = \frac{k_N}{4}(\Delta x)^2\left(u_{xx}(x_i',t)+u_{xx}(x_i'',t)\right)$

Let $N \to \infty$ then $x_{i-1} \to x_i$, $x_{i+1} \to x_i$, $\Delta x \to 0$

Assume $\frac{k_N(\Delta x)^2}{2} \to K$ (diffusivity constant)

Then $\frac{\partial u}{\partial t}(x_i,t) = K\,u_{xx}(x_i,t)$

and $\forall x \in (0,L)$,

$$\frac{\partial u}{\partial t}(x,t) = K\,u_{xx}(x,t).$$

(the one-dimensional heat equation.)

A natural formulation for a heat transfer problem would include

- initial cond. (temperature at every point at $t=0$)
- boundary condition:
  either constant temperature,
  or thermal insulation,
  or other condition for heat influx.)

(IC) $u|_{t=0} = u_0(x)$

(BC) : (1) $u|_{x=0} = T_0$, $u|_{x=L} = T_1$

(first type, given temp.)
Dirichlet

(2) $u_x|_{x=0} = 0$, $u_x|_{x=L} = 0$

(second type, insulation,
Neumann)

(3) $u_x|_{x=0} = -h(T_e - u(0,t))$

$u_x|_{x=L} = h(T_e - u(0,t))$

(third type, Newton's law of cooling, Robin cond.)

Example: Find a time-independent (steady-state) solution of the heat transfer problem

$$\begin{cases} u_t - k u_{xx} = 0 \\ u|_{x=0} = T_0, \quad u|_{x=L} = T_1 \end{cases}$$

$u = u(x)$; $\quad u_{xx} = 0 \Rightarrow \quad u = Ax + B$

$$\Rightarrow u = T_0 + (T_1 - T_0)\frac{x}{L}$$

$$= T_1 \frac{x}{L} + T_0\left(1 - \frac{x}{L}\right)$$

$$= \frac{1}{L}\left(T_1 x + T_0 (L - x)\right)$$

Exercises 0.1.4: 1–3.