Integration by parts for piecewise smooth functions

The integration by parts formula is useful in many instances, in particular in problem 4 (1.3). However, if integrands are discontinuous functions, its formulation has to be modified accordingly.

Lemma (Integration by parts). *Let $f(x)$, $g(x)$ be piecewise smooth on (a,b), with points of discontinuity x_i, $i = 1,\ldots,p$. Then*

$$\int_a^b f(x)\,g'(x)\,dx = \Big[f(x)\,g(x)\Big]_{a+0}^{b-0} - \sum_{i=1}^{p}\Big[f(x)\,g(x)\Big]_{x_i-0}^{x_i+0} - \int_a^b f'(x)\,g(x)\,dx\,.$$

Proof. (See also problems 19, 20 in Section 1.2.) The functions $f(x)$, $g(x)$ are smooth when restricted to each of the intervals (x_i, x_{i+1}), $i = 0,\ldots,p$ (where $x_0 = a$, $x_{p+1} = b$) and have the right and left limits at the end points. The usual integration by parts formula then implies

$$\begin{aligned}\int_{x_i}^{x_{i+1}} (f(x)\,g(x))'\,dx &= \int_{x_i}^{x_{i+1}} f'(x)\,g(x)\,dx + \int_{x_i}^{x_{i+1}} f(x)\,g'(x)\,dx\\ &= [f(x)\,g(x)\Big]_{x_i+0}^{x_{i+1}-0} = -[f(x)\,g(x)\Big]_{x_{i+1}-0}^{x_i+0}\,.\end{aligned}$$

For $i = 0$ this gives $f(x_1 - 0) - f(a+0)$, and for $i = p+1$ we have $f(b-0) - f(x_p + 0)$. Considering these two special cases, and taking the sum over $i = 1,\ldots,p$, we obtain the integration by parts formula in the form that was stated. □