Homework Assignment 8 – Additional problems on the 1D wave equation

Due for the midterm on Tuesday, December 4, 2012

Problems 1 and 2 provide an alternative way to justify the use of D'Alembert's formulas for solution of the vibrating string problem:

1. Let $f(x)$, defined for $-\infty < x < \infty$ be smooth, $2L$-periodic and odd. Show that the following are true:

   (a) $f(0) = f(L) = 0$

   (b) The D'Alembert solution defined by

   $$u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$$

   is a solution of the problem

   $$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L \\ &u(0,t) = 0, \quad u(L,t) = 0 \\ &u(x,0) = f(x), \quad u_t(x,0) = 0 \end{aligned}$$

   [You may use the fact that the D'Alembert solution satisfies the wave equation for $-\infty < x,\, t < \infty$, as well as the initial conditions. It remains to show that the boundary conditions are satisfied for all t.]

2. Let $g(x)$, defined for $-\infty < x < \infty$ be smooth, $2L$-periodic and even. Show that the following are true:

   (a) $g'(0) = g'(L) = 0$

   (b) The D'Alembert solution defined by

   $$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\, dz$$

   is a solution of the problem

   $$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L \\ &u_x(0,t) = 0, \quad u_x(L,t) = 0 \\ &u(x,0) = 0, \quad u_t(x,0) = g(x). \end{aligned}$$

In practice we can apply D'Alembert's formulas even for intial data which are not smooth; in this case they can be used to define "generalized solutions" of the wave equation.

3. Solve $u_{tt} - u_{xx} = 0$, $-\infty < x, t < \infty$, with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$:

(a) $f(x) = \sin x$, $g(x) = \cos x$

(b) $f(x) = \ln(1 + x^2)$, $g(x) = 4 + x$

4. For the problem

$$u_{tt} - u_{xx} = 0, \quad -\infty < x, t < \infty$$
$$u(x, 0) = 0, \quad u_t(x, 0) = g(x)$$

where

$$g(x) = 1, \text{ for } |x| < a, \quad \text{and } g(x) = 0 \text{ for } |x| \geq a,$$

plot snapshots of the solutions $u(x, t)$ for $t = a/2$, $t = a$, $t = 3a/2$, $t = 2a$, $t = 5a/2$.

[Hint: D'Alembert's formula gives us

$$u(x, t) = \frac{1}{2} \int_{x-ct}^{x+ct} g(z)\, dz = \frac{1}{2}\, (\textit{length of}\, (x - ct, x + ct) \cap (-a, a)).$$

If $t = a/2$, this takes on different values if $|x| < a/2$, $a/2 < |x| < 3a/2$ and for $x > 3a/2$. Continue in this manner for each case.]

5. For the problem

$$u_{tt} - u_{xx} = 0, \quad 0 < x < L$$
$$u(0, t) = u(L, t) = 0$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

sketch a diagram in the (x, t) plane showing the values of the solution for $0 < x < L$, $t > 0$:

(a) $f(x) = 1$ for $L/4 < |x| < 3L/4$, and $f(x) = 0$ otherwise,

(b) $f(x) = 1$ for $0 < |x| < L$.

[Hint: the solution will assume only finitely many values, in different regions of the (x, t) plane.]

6. Sketch the same type of diagram as in the previous problem for the initial data from part (a) in the case of the wave equation with the boundary conditions

$$u_x(0, t) = u_x(L, t) = 0.$$