Homework Assignment 7 – Additional problems

Quiz on Tuesday, November 6, 2012

1. Consider the Sturm-Liouville problem

$$\begin{cases} \varphi'' + \lambda\,\varphi = 0, \quad 0 < x < L, \\ + \text{ two-point boundary conditions}, \end{cases}$$

where we allow $\varphi(x)$ to be complex-valued and $\lambda \in \mathbb{C}$. The boundary conditions are any, such that

$$\big[\,\varphi'\,\psi\,\big]_{x=0}^{x=L} = \big[\,\varphi\,\psi'\,\big]_{x=0}^{x=L}$$

for any φ and ψ that satisfy them. Show that then necessarily $\mathrm{Im}(\lambda) = 0$, so in fact $\lambda \in \mathbb{R}$. [Hint: multiply the equation by $\bar{\varphi}$, integrate over $(0, L)$ and use the boundary conditions.]

2. A natural boundary condition for the heat equation is the "Newton's cooling law" condition

$$\vec{q}(\vec{x}, t) \cdot \vec{n}(\vec{x}) = k(u(\vec{x}, t) - T_e), \tag{1}$$

where $\vec{n}(\vec{x})$ is the exterior unit normal on the boundary, $\vec{x} = (x, y, z)$, T_e is the exterior temperaure, and $k > 0$ is a constant which characterizes the rate of thermal exchange at the boundary.

   (a) Consider the infinite slab determined by conditions $0 < z < L$. Show that boundary conditions (1) take the form

$$\begin{aligned} u_z(0, t) &= h(u(0, t) - T_e) \\ -u_z(L, t) &= h(u(L, t) - T_e), \end{aligned}$$

   where $h > 0$ is a constant.

   (b) If $T_e = 0$ show that the boundary conditions from (a) take the form

$$\begin{aligned} &\cos\alpha\, u(0, t) - L \sin\alpha\, u_z(0, t) = 0 \\ &\cos\beta\, u(L, t) + L \sin\beta\, u_z(L, t) = 0, \end{aligned}$$

   where $0 < \alpha, \beta < \pi/2$.

   (c) Let $T_e = 0$, and let $u = u(z)$ be time-independent and satisfy boundary conditions from (a). Verify that the condition of nonnegativity of eigenvalues of the Sturm-Liouville problem, $\big[\,s(z)\,u(z)\,u'(z)\,\big]_{z=0}^{z=L} \le 0$ holds in this case.

3. Consider the Sturm-Liouville problem

$$\begin{cases} \varphi'' + \lambda\,\varphi = 0, \quad 0 < x < L, \\ \varphi(0) = 0, \quad \varphi(L) - h\,\varphi'(L) = 0, \end{cases}$$

where $h > 0$ is a constant. Show that the problem has only positive eigenvalues if $h \geq L$ and exactly one negative eigenvalue if $h < L$. [This shows that the condition $\left[s\,\varphi\,\varphi' \right]_{x=0}^{x=L} \leq 0$ is sufficient but not necessary for the eigenvalues of a Sturm-Liouville problem to be nonnegative.]

4. Use the method of separation of variables to find the solution of the problem

$$
\begin{aligned}
&u_t - K u_{xx} = 0, \quad 0 < x < L, \quad t > 0. \\
&u(0,t) = 0, \quad u(L,t) = h\, u_x(L,t) \\
&u(x,0) = u_0(x),
\end{aligned}
$$

where $h > 0$ and $u_0(x)$ is a given smooth function. Show that solutions of the problem are generally not transient: find $u_0(x)$ such that $|u(x,t)| \to \infty$ as $t \to \infty$. Formulate conditions on $u_0(x)$ under which solutions $u(x,t)$ *are* transient.