Thu, Aug. 11, 2011 **MATH 462, Midterm 2** Prof. V. Panferov

Name: (print) ______________________________

CSUN ID No. : Solutions.

This test includes 6 questions in the main part (48 points in total) and one bonus question worth an extra 6 points. Please check that your copy of the test has 7 pages. The duration of the test is 60 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Compute the determinant:

$$D = \begin{vmatrix} 1 & n+1 & \dots & n^2-n+1 \\ 2 & n+2 & \dots & n^2-n+2 \\ \vdots & \vdots & \dots & \vdots \\ n & 2n & \dots & n^2 \end{vmatrix}$$

(all integers from 1 to n^2 arranged sequentially into columns of an $n \times n$ matrix).

2 cases: $n=2$: $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$;

$n \geq 3$: subtract the first column from the remaining ones:

$$D = \begin{vmatrix} 1 & n & 2n & & n(n-1) \\ 2 & n & 2n & & n(n-1) \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & 2n & \dots & n(n-1) \end{vmatrix} = 0$$ since columns 2, 3... are multiples of each other.

OR: reduce by subtracting the 1st row:

$$D = \begin{vmatrix} 1 & n+1 & 2n+1 & \dots & n(n-1)+1 \\ 1 & 1 & 1 & & 1 \\ 2 & 2 & 2 & & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ n-1 & n-1 & n-1 & \dots & n-1 \end{vmatrix} = 0$$

since rows 2, 3 ... are multiples of each other.

2. (8 points) Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Suppose that β is a basis of V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis of W.

"$\Rightarrow$" Assume T is an isomorphism.
Then $\dim(V) = \dim(W) = n$.
and T is onto $\Rightarrow$ $\operatorname{span}(T(\beta)) = R(T) = W$.
Then $T(\beta)$ is a generating set
with n vectors $\Rightarrow$ $T(\beta)$ is a basis.

"$\Leftarrow$" Assume that $T(\beta)$ is a basis of W.
Then $\operatorname{span}(T(\beta)) = W \Rightarrow T$ is onto.
$T(\beta)$ is linearly independent
$\Rightarrow$ $c_1 T(v_1) + \dots + c_n T(v_n) = 0$
implies $c_1 = \dots = c_n = 0$
if $\beta = (v_1, \dots v_n)$
Let $x \in N(T)$, then $x = c_1 v_1 + \dots + c_n v_n$
$\Rightarrow T(x) = T(c_1 v_1 + \dots + c_n v_n)$
$= c_1 T(v_1) + \dots + c_n T(v_n) = 0$
$\Rightarrow c_1 = \dots = c_n = 0$
$\Rightarrow x = 0$
So $N(T) = \{0\}$ $\Rightarrow$ T is one-to-one.

Continued...

3. (10 points) (a) Let V be a finite-dimensional vector space and let $T : V \to V$ be linear. If $\text{rank}(T^2) = \text{rank}(T)$ prove that $V = R(T) \oplus N(T)$.

Step 1: $N(T^2) = N(T)$.

Since $N(T) \subseteq N(T^2)$, and $\dim(N(T)) + \dim(R(T)) = \dim(V)$

$\dim(N(T^2)) + \dim(R(T^2)) = \dim(V)$

$\Rightarrow \dim(N(T)) = \dim(N(T^2))$

$\Rightarrow N(T) = N(T^2)$ (since $N(T) \subseteq N(T^2)$).

Step 2: $R(T) \cap N(T) = \{0\}$.

Let $y \in R(T) \cap N(T)$. Then $y = T(x)$ for some x and $T(y) = 0$.

Then $T(T(x)) = T^2(x) = 0 \Rightarrow x \in N(T^2)$

$\Rightarrow x \in N(T) \Rightarrow T(x) = 0 \Rightarrow y = 0$.

Step 3: By rank-nullity Thm, $\dim(R(T)) + \dim(N(T)) = \dim(V)$.

Since $R(T) \cap N(T) = \{0\}$,

by the properties of direct sums

$V = R(T) \oplus N(T)$.

(b) Give an example of $T : \mathbb{R}^2 \to \mathbb{R}^2$, with a nontrivial nullspace, such that $V = R(T) \oplus N(T)$.

Example: Projection along any subspace:

$T_A(x) = Ax$, where

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, or $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,

... etc.

a scalar multiple of a projection, such as

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ also works.

Continued...

4. (8 points) If A and B are $n \times n$ matrices prove that $\det(AB) = \det(A)\det(B)$.

Let $A = E_1 \dots E_k U$ where U is upper triangular (for example, rref(A)) and $E_1 \dots E_k$ - elementary matrices.

<u>2 cases</u>: (i) $\det(A) = 0 \Rightarrow A$ is not invertible $\Rightarrow U$ has at least one zero row

Then
$$\det(AB) = \det(E_1) \dots \det(E_k) \det(UB) = 0,$$
since elementary operations either preserve the determinant, or multiply by -1, or $k \neq 0$, and $\det(UB) = 0$ since UB has at least one zero row.

(ii) $\det(A) \neq 0$ then U can be chosen as I (identity of size n)

Then
$$\det(AB) = \det(E_1) \dots \det(E_k) \det(B).$$
However, $\det(A) = \det(E_1) \dots \det(E_k) \det(I)$

Since every elementary operation either leaves the determinant the same, or multiplies by -1, or $k \neq 0$.

5. (8 points) Define $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ by $T(A) = A^t$. Diagonalize T by finding a basis β and a diagonal matrix D such that $[T]_\beta = D$.

One approach: Take $\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ – a basis of $M_{2\times 2}(F)$.

In this basis

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Char. poly $f(t) = -(1-t)^3(1+t)$

Eigenvalues $\lambda = 1, 1, 1, -1$

Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$$\Rightarrow D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

Another approach: Solve

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow$ either $\lambda = \pm 1$ or $a, b, c, d = 0$.

If $\lambda = 1$, $b = c$, a, d arbitrary $\Rightarrow$

3 lin. indep. soln: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(basis for symmetric matrices).

If $\lambda = -1$, $a = d = 0$, $b = -c$ $\Rightarrow$

1 lin. indep. soln: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Found 4 lin. indep. eigenvectors for eigenvalues $\lambda = 1, -1$ $\Rightarrow$ basis of eigenvectors.

Continued...

6. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

   Notations V and W are used for vector spaces over a field F; T denotes a linear transformation, and A and B denote matrices.

   (a) If $[T]_\beta = I$ (the $n \times n$ identity matrix) for some basis β then T is the identity operator.

   (b) If V is a vector space over $\mathbb{C}$ then every linear operator $T : V \to V$ has at least one eigenvalue.

   (c) $AB = I$ implies that A and B are invertible.

   (d) The matrices A and B are similar if $B = Q^t AQ$.

   (e) The determinant of a lower triangular matrix is a product of its diagonal entries.

   (f) If U is an upper triangular matrix then the diagonal entries are eigenvalues of U.

   (g) Similar matrices always have the same characteristic polynomials.

   (h) The sum of two eigenvectors of an operator T is always an eigenvector of T.

Answers:

(a) T (T is determined by its values on a basis.)

(b) T (every characteristic polynomial splits.)

(c) F (not unless $n = m$)

(d) F (Q^{-1}, not Q^t.)

(e) T (same as for upper-triangular.)

(f) T

(g) T (since $A \sim B \Rightarrow A - tI \sim B - tI$)

(h) F (not if they correspond to distinct eigenvalues.)

7. (bonus: 6 points) Let T be a linear operator on an n-dimensional vector space V, with the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

Find a vector $v \in V$ such that the set $\{v, T(v), \dots, T^{n-1}(v)\}$ is a basis of V.

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{pmatrix}$$

any vector V with $x_n \neq 0$ will do.

For instance, if $V = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ then

$Ae_n = e_{n-1}$, $A^2 e_n = e_{n-2} \dots A^{n-1} e_n = e_1$

— basis.