Thu, July 28, 2011 **MATH 462, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

CSUN ID No. : ____________________

This test includes 6 questions in the main part (44 points in total) and one bonus question worth an extra 6 points. Please check that your copy of the test has 7 pages. The duration of the test is 60 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Let V be a vector space over a field F. Prove, using the axioms of a vector space that for every x in V we have $(-x) = (-1)x$.

$$
\begin{aligned}
(-1)x &= (-1)x + 0 = (-1)x + (x + (-x)) \\
&= ((-1)x + x) + (-x) \\
&= (-1+1)\cdot x + (-x) = 0.x + (-x) \\
&= (0.x + 0) + (-x) \\
&= (0.x + (x + (-x))) + (-x) \\
&= ((0.x + x) + (-x)) + (-x) \\
&= ((0.x + 1.x) + (-x)) + (-x) \\
&= ((0+1)x + (-x)) + (-x) \\
&= (x + (-x)) + (-x) \\
&= 0 + (-x) = (-x).
\end{aligned}
$$

2. (8 points) Let V be a vector space of dimension n.

(a) If S is a linearly independent set in V that contains exactly n vectors, prove that S is a basis of V.

By contradiction: if S is not a basis
then S is not a generating set $\Rightarrow$
$\Rightarrow \exists\, v \in V \,:\, v \notin \mathrm{span}(S)$
$\Rightarrow S \cup \{v\}$ is a linearly indep. set with $n+1$ vectors,
a contradiction with the replacement theorem.

(b) If G is a generating set in V that contains exactly n vectors, prove that G is a basis of V.

By contradiction: if G is not a basis
then G is linearly dependent
$\Rightarrow$ one of the vectors in G is a linear combination of the remaining $n-1$ vectors
$\Rightarrow$ the remaining $n-1$ vectors form a generating set,
a contradiction with the replacement theorem.

3. (8 points) Let W_1 and W_2 be subspaces of a vector space V. Prove that V is a direct sum of W_1 and W_2 if and only if each vector in V can be represented uniquely as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

"$\Rightarrow$" Assume $V = W_1 \oplus W_2$.

then $\forall v \in V \ \exists \ x_1 \in W_1, x_2 \in W_2$:

$$v = x_1 + x_2 \quad \text{since } V = W_1 + W_2.$$

Assume that $v = x_1 + x_2 = x_1' + x_2'$ – another representation

Then $W_1 \ni x_1 - x_1' = x_2' - x_2 \in W_2$

$$\Rightarrow x_1 - x_1', \ x_2' - x_2 \in W_1 \cap W_2 = \{0\}.$$

$$\Rightarrow x_1 = x_1', \ x_2 = x_2'.$$

"$\Leftarrow$" Assume that $\forall v \in V \ \exists \ x_1 \in W_1, x_2 \in W_2$:

$$v = x_1 + x_2$$

and that $x_1' + x_2' = x_1 + x_2$ implies $x_1' = x_1$, $x_2' = x_2$.

Then (i) $V = W_1 + W_2$,

If $x \in W_1 \cap W_2$ then

$$x = \underset{\substack{\uparrow \\ W_1}}{0} + \underset{\substack{\uparrow \\ W_2}}{x} = \underset{\substack{\uparrow \\ W_1}}{x} + \underset{\substack{\uparrow \\ W_2}}{0}$$

$$\Rightarrow x = 0$$

$$\Rightarrow W_1 \cap W_2 = \{0\}.$$

Continued...

4. (8 points) Consider the linear transformation $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find bases for $R(T)$ and $N(T)$ and determine the rank and the nullity of T.

$$T(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow R(T) = \operatorname{span}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$= \operatorname{span}\Big(\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{basis.}} \Big)$$

$$\operatorname{rank}(T) = \dim(R(T)) = 2.$$

$$\operatorname{nullity}(T) = \dim(P_2(\mathbb{R})) - \operatorname{rank}(T)$$

$$= 3 - 2 = 1.$$

If $f(x) = a + bx + cx^2$

$$f(1) - f(2) = -b - 3c = 0 \Rightarrow b = -3c$$

$$f(0) = a = 0$$

$$\Rightarrow f(x) = c(x^2 - 3x)$$

$$N(T) = \operatorname{span}(x^2 - 3x)$$

5. (6 points) Define $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ by $T(A) = A^t$. Compute the matrix of T in the standard basis of $M_{2\times 2}(F)$:

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Continued...

6. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

Notations V and W are used for vector spaces over a field F.

(a) If a vector space has finite basis then the number of vectors in any basis is the same.

(b) The empty set is a subspace of any vector space.

(c) If W_1 and W_2 are subspaces of V and $\dim(W_1) < \dim(W_2)$ then $W_1 \subseteq W_2$.

(d) $T : V \to W$ is linear if and only if $\forall\, x, y \in V\ \forall\, a \in F$, $T(x + ay) = T(x) + aT(y)$.

(e) The dimension of $M_{m\times n}(F)$ is $m + n$.

(f) If $T \in \mathscr{L}(V;W)$ then $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(W)$.

(g) If T and U are linear transformations and $[T]_\beta^\gamma = [U]_\beta^\gamma$ then $T = U$.

(h) If $T \in \mathscr{L}(V;W)$ is onto, then the image of any linearly independent set in V is linearly independent in W.

Answers:

(a) T (corollary of replacement theorem.)

(b) F (any subspace has to contain at least the zero vector.)

(c) F (take a line and a plane in $\mathbb{R}^3$ such that a line is not contained in the plane)

(d) T

(e) F (dim $M_{m\times n}(F) = mn$)

(f) F (rank (T) + nullity $(T) = \dim(V)$.)

(g) T (linear transf. is determined fully by its values on a basis.)

(h) F (not unless T is one-to-one.)

7. (bonus: 6 points) Let V and W be vector spaces such that $\dim(V) = \dim(W)$ and let $T : V \to W$ be linear. Show that there exists ordered bases β and γ such that $[T]_\beta^\gamma$ is a diagonal matrix.

Let $\beta = (v_1 \ldots v_n)$

and $\gamma = T(\beta)$.

then $[T]_\beta^\gamma$ is the $n \times n$ identity matrix.