Mon, Nov 21, 2011 **MATH 462, Midterm 2** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test includes 6 questions in the main part (46 points in total) and one bonus question worth an extra 6 points. Please check that your copy of the test has 7 pages. The duration of the test is 75 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Suppose that β is a basis of V. If $T(\beta)$ is a basis of W prove that T is an isomorphism.

$\beta = (u_1, u_2, \ldots u_n)$, $\dim(V) = n$.

$T(\beta) = (T(u_1), T(u_2), \ldots T(u_n))$ – basis of W.

$\dim(W) = n$.

Since $R(T) = \text{span}(T(\beta)) = W$,

T is onto.

Since $\dim(V) = \dim(W)$,

onto $\Leftrightarrow$ one-to-one $\Leftrightarrow$ invertible

$\Rightarrow$ T is an isomorphism.

2. (8 points) Compute the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}.$$

$$f(t) = \begin{vmatrix} -t & 0 & \dots & 0 & -a_0 \\ 1 & -t & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \\ 0 & \dots & \dots & 1 & -a_{n-1}-t \end{vmatrix}$$

Method 1: If $t \neq 0$, expand along the first row $\Rightarrow f(0) = (-1)^n a_0$. If $t \neq 0$, use Gauss elimination to bring to upper triangular form.

Method 2: Use expansion along the first row and induction.

Method 3: Use expansion along the last column

Answer:

$$f(t) = (-1)^n (a_0 + a_1 t + \dots + a_{n-1} t^{n-1} + t^n).$$

3. (8 points) Suppose V is a vector space of dimension n and and T is a linear operator with n distinct eigenvalues. Show that V can then be represented as a direct sum of one-dimensional T-invariant subspaces.

If T has n distinct eigenvalues then T is diagonalizable and has a basis of eigenvectors $\beta = (u_1 \dots u_n)$,

$\forall j \quad T(u_j) = \lambda_j \cdot u_j$

$\forall j \quad E_{\lambda_j} = \{x : T(x) = \lambda_j x\} = span\{u_j\}$ is T-invariant and one-dimensional.

(indeed $x \in E_{\lambda_j} \Rightarrow T(x) = \lambda_j \cdot x$
$\Rightarrow T(T(x)) = \lambda_j \cdot T(x)$
$\Rightarrow T(x) \in E_{\lambda_j}$)

I claim that $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_n}$.

Indeed, β - basis $\Rightarrow$

$\forall x \in V \quad x = a_1 u_1 + \dots + a_n u_n$
$= v_1 + \dots + v_n$, where $v_j \in E_{\lambda_j}$

$\Rightarrow V = E_{\lambda_1} + \dots + E_{\lambda_n}$.

Also if $x \in E_{\lambda_i} \cap \sum_{j \neq i} E_{\lambda_j}$ then

$x = c_i u_i = c_1 u_1 + \dots + c_{i-1} u_{i-1} + c_{i+1} u_{i+1} \dots + c_n u_n$

$\Rightarrow c_1 u_1 + \dots + c_{i-1} u_{i-1} - c_i u_i + c_{i+1} u_{i+1} + \dots + c_n u_n = 0$

$\Rightarrow \forall j \; c_j = 0$ since β is a basis

$\Rightarrow x = 0$.

Continued...

$\Rightarrow \forall i \quad E_{\lambda_i} \cap \sum_{j \neq i} E_{\lambda_j} = \{0\}$.

4. (8 points) Give a proof of the statement that if $B, A \in M_{n\times n}(F)$ and B is obtained from A by interchanging any two rows, then $\det(B) = -\det(A)$. [You may use the definitions and other theorems concerning determinants.]

Let a_j be rows of A:

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix}; \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ a_i \\ \vdots \\ a_n \end{pmatrix}$$

Then

$$0 = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i + a_j \\ \vdots \\ a_j + a_i \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ a_i \\ a_j + a_i \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ a_j \\ a_j + a_i \\ a_n \end{pmatrix}.$$

(two identical rows.)

$$= \det \begin{pmatrix} a_1 \\ a_i \\ a_j \\ a_n \end{pmatrix} + \underbrace{\det \begin{pmatrix} a_1 \\ a_i \\ a_i \\ a_n \end{pmatrix}}_{=0 \text{ (two identical rows)}}$$

$$+ \underbrace{\det \begin{pmatrix} a_1 \\ a_j \\ a_j \\ a_n \end{pmatrix}}_{=0 \text{ (two identical rows.)}} + \det \begin{pmatrix} a_1 \\ a_j \\ a_i \\ a_n \end{pmatrix}$$

$$= \det(A) + \det(B)$$

$$\Rightarrow \det(B) = -\det(A).$$

5. (8 points) Define $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f(x)) = f(x) + f(2)x$. Diagonalize T by finding a basis β and a diagonal matrix D such that $[T]_\beta = D$.

Let $\alpha = (1, x, x^2)$. Then $T(\alpha) = (1+x, 3x, x^2+4x)$

$$\Rightarrow \quad [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f(t) = (1-t)^2(3-t) \Rightarrow \lambda = 1, \lambda = 3$$

– eigenvalues.

$\lambda = 3 \Rightarrow$ Eigenvector $V_1 = X$ (Guess since $T(x) = 3x$, from the computation of the basis.)

$\lambda = 1 \Rightarrow$

$$[T - \lambda]_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Eigenvectors:

$V_2 = 2 - x$; $\quad V_3 = 2x - x^2$

$\beta = (x, 2-x, 2x-x^2)$.

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Continued...

6. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

   Notations V and W are used for vector spaces over a field F; T denotes a linear transformation, and A and B denote matrices.

   (a) If A and B are similar matrices and A is invertible then B is invertible.

   (b) A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if the algebraic multiplicity of every eigenvalue λ equals the dimension of E_λ.

   (c) If V is a vector space over $\mathbb{C}$ then every linear operator $T : V \to V$ has at least one eigenvalue.

   (d) $M_{2\times3}(F)$ is isomorphic to F^6.

   (e) There exists a square matrix with no eigenvectors.

   (f) Every polynomial of degree n with the leading coefficient $(-1)^n$ is a characteristic polynomial of some linear operator.

   (g) The sum of two eigenvectors of an operator T is always an eigenvector of T.

   (h) If T is a linear operator on a finite-dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

Answers:

(a) T (same char. polynomials, too)

(b) F (char. poly. has to split.)

(c) F (if V is infinite-dimensional; example is a shift operator on the space of sequences; Answer is T if finite-dim.

(d) T

(e) T

(f) T (see problem 2)

(g) F

(h) F

7. (bonus: 6 points) If B and C are matrices in $M_{n\times n}(F)$, is it always true that BC and CB are similar? Does it make any difference whether B and C are invertible?

No, it is not always true. Take for instance two matrices B, C such that $CB = 0$, but $BC \neq 0$.

Example: $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

The statement is true if at least one of the matrices is invertible:

if $\exists B^{-1}$, then

$$B^{-1}(BC)B = (B^{-1}B)(CB) = CB$$

$\Rightarrow BC$ is similar to CB.

Similarly if $\exists C^{-1}$ then

$C^{-1}(CB)C = BC \Rightarrow BC$ is similar to CB.