Wed, Oct 5, 2011 **MATH 462, Midterm 1** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions

This test includes 7 questions (54 points in total) in the main part, and one bonus question worth an extra 6 points. Please check that your copy of the test has 8 pages. The duration of the test is 1 hour 15 minutes.

**Your scores:** (do not enter answers here)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | total |
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**Important:** The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Show based on the Replacement Thorem that if $V$ is a vector space of dimension $n$ then any linearly independent set in $V$ that has exactly $n$ vectors is a basis of $V$.

Let $V$ be a vector space over $F$, $\dim(V) = n$.

Let $S = \{v_1, \dots v_n\}$ – linearly indep.

If $S$ is not a basis then $S$ is not generating $V$,

$\exists\, w \in V$, $w \notin \operatorname{span}\{S\}$

Then $\{w\} \cup S$ is a linearly indep. set with $n+1$ elements,

in contradiction to the Replacement Theorem.

2. (8 points) Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2),\ (b_1, b_2) \in V$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$c \cdot (a_1, a_2) = (ca_1, a_2/c) \quad \text{if } c \neq 0; \qquad 0 \cdot (a_1, a_2) = (0, 0).$$

Is $V$ a vector space over $\mathbb{R}$ with the operations '+' and '$\cdot$'? Justify your answer.

No. If $c_1, c_2 \neq 0$,

$$(c_1+c_2)(a_1,a_2) = \left((c_1+c_2)a_1, \frac{a_2}{c_1+c_2}\right)$$

$$c_1(a_1,a_2) + c_2(a_1,a_2) = \left(c_1a_1, \frac{a_2}{c_1}\right) + \left(c_2a_1, \frac{a_2}{c_2}\right)$$
$$= \left(c_1a_1 + c_2a_1, \frac{a_2}{c_1} + \frac{a_2}{c_2}\right)$$
$$= \left((c_1+c_2)a_1, \left(\frac{1}{c_1} + \frac{1}{c_2}\right)a_2\right)$$

Take $c_1, c_2$ so that

$$\frac{1}{c_1+c_2} \neq \frac{1}{c_1} + \frac{1}{c_2},$$ for instance $c_1 = 1$, $c_2 = 1$

$\Rightarrow$ the distributive law is violated.

*Continued...*

3. (6 points) The set of solutions of the linear system

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ ix_1 + ix_2 - x_3 - x_4 = 0 \end{cases}$$

is a subspace of $\mathbb{C}^4$. Find a basis of this subspace.

Matrix form:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ i & i & -1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1-i & -1-i & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Choose $x_2 = s$, $x_4 = t$ (free)

then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The vectors are lin. indep (by 0-1 principle)

$$\Rightarrow \beta = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$ is a basis

Continued...

4. (8 points) Let $W$ be the subspace of $M_{2\times 2}(\mathbb{C})$ consisting of all symmetric matrices. Construct an isomorphism from $W$ to $\mathbb{C}^3$. Justify your answer.

$$W = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} ; \quad a, b, c \in \mathbb{C} \right\}$$

Let $T : \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then $T$ is linear (matrices are added component-by-component, and so are vectors in $\mathbb{C}^3$. Likewise for scalar multiplication)

The inverse of $T$ is

$$T^{-1} : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ b & c \end{pmatrix};$$

then $T^{-1} \circ T = I_W$;

$T \circ T^{-1} = I_{\mathbb{C}^3}$

$\Rightarrow$ $T$ is invertible.

5. (8 points) Define $T : P_2(F) \to P_2(F)$ by $T(f(x)) = f(x) + f'(x)$.

(a) Find the matrix of the transformation $T$ is the standard basis of $P_2(F)$: $\beta = (1, x, x^2)$.

$T(1) = 1$ ; $T(x) = x + 1$ , $T(x^2) = x^2 + 2x$

$$\Rightarrow [T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\text{rank}\left([T]_\beta^\beta\right) = 3 \Rightarrow \text{nullity} = 0$

$\Rightarrow$ there are no polynomials $f(x)$ besides zero that are transformed into zero by $T$

$\Rightarrow N(T) = \{0\}$.

then $R(T) = P_2(F)$.

(b) Find bases of $R(T)$ and $N(T)$. [Hint: a basis of a vector space $V$ is a subset of $V$.]

basis of $R(T)$ : $\{1, x, x^2\}$

basis of $N(T)$ : $\varnothing$.

6. (10 points) Formulate and prove the Rank-Nullity Theorem.

Let $V, W$ be vector spaces over $F$ and let $T: V \to W$ be linear.
If $V$ is finite-dimensional, then
$$\dim(R(T)) + \dim(N(T)) = \dim(V).$$

<u>Proof</u>: Let $\beta_1 = \{u_1 \dots u_k\}$ be a basis of $N(T)$
($N(T)$ is finite-dim, since it is a subspace of $V$)

Extend $\beta_1$ to $\beta = \{u_1 \dots u_k, u_{k+1}, \dots u_n\}$ - basis of $V$.

Claim: $\beta_2 = \{T(u_{k+1}) \dots T(u_n)\}$ - basis of $R(T)$.

Indeed, if $y \in R(T)$ then $\exists\, x = a_1 u_1 + \dots + a_k u_k + a_{k+1} u_{k+1} + \dots + a_n u_n$

such that
$$y = T(x) = T(\underbrace{a_1 u_1 + \dots + a_k u_k}_{\in N(T)} + a_{k+1} u_{k+1} + \dots + a_n u_n)$$
$$= a_{k+1} T(u_{k+1}) + \dots + a_n T(u_n)$$

$\Rightarrow \beta_2$ generates $R(T)$.

Check that $\beta_2$ is lin. indep.:

if $c_{k+1} T(u_{k+1}) + \dots + c_n T(u_n) = 0$

then $T(c_{k+1} u_{k+1} + \dots + c_n u_n) = 0$

$\Rightarrow c_{k+1} u_{k+1} + \dots + c_n u_n \in N(T)$

$\Rightarrow c_{k+1} u_{k+1} + \dots + c_n u_n = -c_1 u_1 - \dots - c_k u_k$ for certain $c_1, \dots c_k$.

$\Rightarrow c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$

$\Rightarrow$ all $c_i = 0$ since $\{u_1 \dots u_n\}$ is a basis of $V$.

Continued...

7. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

   Notations $V$ and $W$ are used for vector spaces over a field $F$. $I_n$ is the identity matrix of size $n$.

   (a) The intersection of any finite number of subspaces in a vector space $V$ is a subspace of $V$.

   (b) The empty set is a subspace of every vector space.

   (c) If $T, U : V \to W$ are both linear and agree on a basis of $V$, then $T = U$.

   (d) If $T, U : V \to W$ are both linear and $[T]_\beta^\gamma = [U]_\beta^\gamma$ for some bases $\beta$ and $\gamma$, then $T = U$.

   (e) $T : V \to W$ is linear if $\forall\, x, y \in V$, $\forall\, a \in F$ $T(x + ay) = T(x) + aT(y)$.

   (f) If $A \in M_{n \times n}(\mathbb{R})$ then $A^2 = I_n$ implies that $A = I_n$ or $A = -I_n$.

   (g) If $A \in M_{n \times n}(\mathbb{R})$ then $A^2 = 0$ (the zero matrix) implies that $A = 0$.

   (h) If $T : V \to W$ is linear and onto, then the image of any linearly independent set in $V$ is linearly independent in $W$.

Answers:

(a) T

(b) F

(c) T

(d) T

(e) T

(f) F

(g) F

(h) F

8. (bonus: 6 points) Let

$$W_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Give an example of a subspace $W_2$ in $F^4$ such that $W_1 \oplus W_2 = F^4$.

Let $W_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Then $W_1 \cap W_2 = \{0\}$

since $k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \ell \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow k = \ell = 0$

and $k \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

Also, if $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in F^4$, then

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ b \\ d \\ d \end{pmatrix} + \begin{pmatrix} a-b \\ 0 \\ c-d \\ 0 \end{pmatrix}$$

$$= b \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + (a-b) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (c-d) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

*The end.*