Wed, Nov 9, 2011 MATH 462 Quiz 6

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Let T be an invertible linear operator on a finite-dimensional vector space. Prove that if T is diagonalizable then T^{-1} is diagonalizable.

Since $Tx = \lambda x \iff x = \lambda T^{-1}(x) \iff T^{-1}(x) = \frac{1}{\lambda} x$, any eigenvector of T is also an eigenvector of T^{-1}. Since T has a basis of eigenvectors, so does $T^{-1} \Rightarrow T^{-1}$ is diagonalizable.

OR:

Let α be a basis of V and $A = [T]_\alpha$.

Then $A = SDS^{-1} \Rightarrow A^{-1} = SD^{-1}S^{-1}$

Since D^{-1} is diagonal, $A^{-1} = [T^{-1}]_\alpha$ is diagonalizable $\Rightarrow T^{-1}$ is diagonalizable.

2. For the linear operator T on the vector space V determine whether the subspace W is invariant: $V = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $W = \{A \in V : A^\top = A\}$.

Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in W$, then

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} b & c \\ a & b \end{pmatrix} \notin W$, unless $a = c$.

$\Rightarrow W$ is not T-invariant.

Please turn over...

3. For the linear operator T on the vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix:

$$V = \mathbb{C}^2, \quad \text{and } T \quad \text{is defined by} \quad T\left(\begin{pmatrix} z \\ w \end{pmatrix}\right) = \begin{pmatrix} z + iw \\ iz + w \end{pmatrix}.$$

let $\alpha = (e_1, e_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ - standard basis of $\mathbb{C}^2$.

Then $[T]_\alpha = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = A.$

$f(t) = \begin{vmatrix} 1-t & i \\ i & 1-t \end{vmatrix} = (1-t)^2 + 1 = 0$

$\Rightarrow t = 1+i$ or $1-i$

$\lambda = 1+i \Rightarrow A - \lambda I = \begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \Rightarrow X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- eigenvector

$\lambda = 1-i \Rightarrow A - \lambda I = \begin{pmatrix} i & i \\ i & i \end{pmatrix} \Rightarrow X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- eigenvector.

$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ - basis of eigenvectors;

$[T]_\beta = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}.$