Dec 13, 2014

MATH 462: Final exam review questions

1. Let $\beta = (v_1, v_2, \dots, v_n)$ be a basis of a vector space V. Prove that every vector in V can be represented uniquely as a linear combination of vectors in β.

2. Find a basis $(p_0(x), \dots, p_n(x))$ for the vector space $P^n(\mathbb{F})$ such that each $p_i(x)$ is a polynomial of degree n. Justify your answer.

3. Consider the matrix
$$A(x) = \begin{pmatrix} f_1(x) & a_{12} & \dots & a_{1n} \\ a_{21} & f_2(x) & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & f_n(x) \end{pmatrix},$$
where the diagonal entries $f_1(x)$, ..., $f_n(x)$ are polynomials in $P(\mathbb{F})$, and the off-diagonal entries a_{ij}, $i \neq j$ are scalars in $\mathbb{F}$. Show that $\det(A) \in P(\mathbb{F})$ and
$$\deg A(x) \leq \sum_{i=1}^{n} \deg f_i(x).$$

4. (a) If $A \in M^{n\times n}(\mathbb{F})$ is a matrix with entries a_{ij} and $f(t) = \det(A - tI)$, prove that
$$f(t) = (a_{11} - t)\dots(a_{nn} - t) + q(t),$$
where $q(t)$ is a polynomial of degree at most $n-2$.

   (b) Use part (a) to show that the coefficient at the $(n-1)$-st power of t in the polynomial $f(t)$ is $(-1)^{n-1}\mathrm{tr}(A)$, where $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of the matix A.

5. Let $A, B \in M^{n\times n}(\mathbb{F})$ be $n \times n$ matrices.
   (a) Prove that $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
   (b) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

6. If $A, B \in M^{n\times n}(\mathbb{F})$ are similar matrices, prove that $\mathrm{tr}(A) = \mathrm{tr}(B)$. *Hint: use Problem 4.*

7. Define the operators T and U on the space $P(\mathbb{R})$ of all polynomials in variable x as follows:
$$T(f(x)) = f'(x), \qquad U(f(x)) = xf(x).$$
Is the relation $TU = UT$ valid? Find the operator $TU - UT$.

8. Assuming that T and U are linear operators on a vector space V, such that $TU - UT = I$, prove that
$$T^m U - UT^m = mT^{m-1}, \quad m = 1, 2, \ldots$$

9. Prove that the condition $TU - UT = I$ can never be satisfied for operators T and U on a finite-dimensional vector space. *[Hint: Use Problems 5 and 6.]* Comment: the result of Problem 7 shows that the condition of finite-dimensionality is essential here.

10. Let T is a linear operator on a finite-dimensional vector space V. Prove that $V = R(T) + N(T)$ implies $V = R(T) \oplus N(T)$. Give a counter-example to this statement in the case when V is infinite-dimensional.

11. A linear operator $T : V \to V$ is called a projection if $T^2 = T$. If $T \neq 0$ is a projection show that the range of T is the eigenspace corresponding to the eigenvalue 1.

12. A linear operator $T : V \to V$ is called nilpotent if $T^p = 0$ for certain positive integer p. Prove that if an operator is nilpotent then 0 is its only eigenvalue.

13. For the linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_A(x) = Ax$, $x \in \mathbb{R}^3$, where
$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$
give an example of a one-dimensional and a two-dimensional invariant subspace.

14. Let T be a linear operator on a finite-dimensional vector space V and let W_1 and W_2 be T-invariant subspaces of V such that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$ be a basis for V, such that β_i are bases for W_i, $i = 1, 2$. Prove that
$$[T]_\beta = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \text{where} \quad A_i = [T_{W_i}]_{\beta_i}, \ \ i = 1, 2.$$

15. Let T, U be linear operators on a vector space V, such that $UT = TU$.
   (a) Prove that $R(U)$, $N(U)$ are T-invariant.
   (b) If $W = \{x \in V : \exists p \in \mathbb{N} \ \ (U^p(x) = 0)\}$ then W is T-invariant.

16. Compute the determinant of a $2k \times 2k$ matrix:

$$\begin{pmatrix} n & 0 & \cdots & \cdots & 0 & m \\ 0 & n & \cdots & \cdots & m & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & n & m & \cdots & 0 \\ 0 & \cdots & m & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & m & \cdots & \cdots & n & 0 \\ m & 0 & \cdots & \cdots & 0 & n \end{pmatrix}.$$

17. Let $A = \begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix}$. Find all $k \in \mathbb{R}$ such that $\lim_{n\to\infty} A^n$ is finite.

18. Is there a vector space V and a linear operator T on V such that T has *exactly three* T-invariant subspaces?

19. Let A be the complex matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find the Jordan canonical form of A and a Jordan basis. *Hint:*

$$(A-2)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -3 & 9 \end{pmatrix},$$

$$(A-2)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & -3 & -3 & 9 & -27 \end{pmatrix},$$