Tue, Nov 4, 2014 MATH 462, Quiz 6 Prof. V. Panferov

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Let $T : V \to V$ be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

λ is an e/v of T $\iff$ $\exists\, x \neq 0 : T(x) = \lambda x$

$\iff \exists\, x \neq 0 : T^{-1}T(x) = T^{-1}(\lambda x)$

$\iff \exists\, x \neq 0 : x = \lambda T^{-1}(x)$

Since T is invertible, $\lambda = 0$ is not an eigenvalue $(N(T) = \{0\})$

Thus

$\iff \exists\, x \neq 0 : T^{-1}(x) = \lambda^{-1} x$

$\iff \lambda^{-1}$ is an e/v of T.

2. For the linear operator $T : V \to V$ find a basis for the T-cyclic subspace generated by the vector z:

$$V = P_3(\mathbb{R}), \quad T(f(x)) = f''(x), \qquad z = x^3.$$

$z = x^3$

$T(z) = 6x$

$T^2(z) = 0$

$\langle z \rangle_T = \text{span}\{x^3, 6x\} = \text{span}\{x, x^3\}$

basis: $\{x, x^3\}$.

3. Test the linear operator $T : V \to V$ for diagonalizability; if T is diagonalizable find (a) a basis βconsisting of eigenvectors and (b) the diagonal matrix $[T]_\beta$:

$$V = \mathbb{C}^2; \qquad T(z, w) = (z + iw, iz + w).$$

$T = L_A$ where $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & i \\ i & 1-t \end{vmatrix} = (1-t)^2 + 1 = 0$

$\iff (t-1)^2 = -1 \iff t - 1 = \pm i$

$\iff t = 1 \pm i$

2 distinct eigenvalues $\Rightarrow$ diagonalizable.

Eigenvectors:

$\lambda = 1+i \Rightarrow A - tI = \begin{pmatrix} -i & i \\ i & -i \end{pmatrix}$

$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ - eigenvector

$\lambda = 1-i \Rightarrow A - tI = \begin{pmatrix} i & i \\ i & i \end{pmatrix}$

$\Rightarrow x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ - eigenvector

$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$ - basis of eigenvectors

$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}.$