Thu, Oct 23, 2014 — MATH 462, Quiz 5 — Prof. V. Panferov

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. A matrix $A \in M^{n\times n}(\mathbb{C})$ is skew-symmetric if $A^T = -A$. (a) Prove that if A is skew-symmetric and n is odd then A is not invertible. (b) Give an example of an invertible skew-symmetric matrix in the case when n is even.

$A^T = -A$

$\det A^T = \det(-A)$

$\det(A) = (-1)^n \det(A)$ (use property of transpose and the multi-linear property)

If n is odd

$\det(A) = -\det(A)$

$2\det(A) = 0$

$\Rightarrow \det(A) = 0$

$\Rightarrow A$ is not invertible.

If $n = 2$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an invertible sym. matrix ($\det(A) = 1$).

2. Let $A, B \in M^{n\times n}(\mathbb{F})$ be such that $AB = -BA$. Prove that if n is odd, and $\mathbb{F}$ is not a field of characteristic two, then A or B is not invertible.

$\det(AB) = \det(-BA)$ multi-lin, product

$\det(A)\det(B) = (-1)^n \det(B)\det(A)$

$(1-(-1)^n)\det(A)\det(B) = 0$ n - odd

$(1-(-1))\det(A)\det(B) = 0$

$\underbrace{(1+1)}_{\neq 0 \text{ iff } \mathbb{F} \text{ is not of char. } 2}\det(A)\det(B) = 0$

$\det(A)\det(B) = 0 \Rightarrow \det(A) = 0$ or $\det(B) = 0$ at least one of A, B is not invertible

Please turn over...

3. Suppose that $M \in M^{n\times n}(\mathbb{F})$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

where A is a square matrix of size k, and I is the identity matrix of size $n-k$. Prove that $\det(M) = \det(A)$.

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1k} & b_{11} & \cdots & b_{1\,n-k} \\ \vdots & & \vdots & & & \\ a_{k1} & \cdots & a_{kk} & b_{k1} & \cdots & b_{k\,n-k} \\ & & & 1 & & \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & & 1 \end{pmatrix}$$

If $n-k=1$, expand on the last row

$\Rightarrow \det(M) = (-1)^{(k+1)+(k+1)} \det(A) = \det(A)$.

By induction on $n-k$.

If $n-k=m$, $\forall B \in M^{k\times m}$ $\det(M)=\det(A)$

then $\forall B \in M^{k\times m+1}$

$$\det(M) = (-1)^{(k+m+1)+(k+m+1)} \det(\underbrace{M_{k+m+1,\,k+m+1}}_{k+m\times k+m \text{ sub-matrix}})$$

$$= \det(A)$$