Thu, Sept 25, 2014 **MATH 462, Quiz 3** Prof. V. Panferov

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Let $T : M^{2\times3}(\mathbb{F}) \to M^{2\times2}(\mathbb{F})$ be defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

Find the bases for $N(T)$ and $R(T)$.

$N(T)$: $\begin{cases} 2a_{11} - a_{12} = 0 \\ a_{13} + 2a_{12} = 0 \end{cases} \Rightarrow$

$$a_{11} = \frac{t}{2}, \quad a_{12} = t, \quad a_{13} = -2t, \quad a_{21} = u, \quad a_{22} = v, \quad a_{23} = w$$

free

Basis for $N(T)$: $\left\{ \begin{pmatrix} \frac{1}{2} & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$R(T) = \text{span}\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(Could also used dimension theorem:
$\dim(N(T)) = 4 \Rightarrow \dim(R(T)) = 2$
$\Rightarrow$ basis must have 2 vectors,
$\therefore$ pick 2 linearly indep. elements in the range.)

Please turn over...

2. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = 0$ (the zero transformation) if and only if $R(T) \subseteq N(T)$.

"$\Rightarrow$" $\quad T^2 = 0$

Assume $y \in R(T) \Rightarrow y = T(x) \Rightarrow T(y) = T^2(x) = 0$

$\Rightarrow y \in N(T)$.

$\therefore \quad R(T) \subseteq N(T)$.

"$\Leftarrow$" $\quad R(T) \subseteq N(T)$

$\forall x \in V \quad T^2(x) = T(T(x)) = 0$,

since $T(x) \in R(T)$.

3. Define $T : M^{2\times 2}(\mathbb{F}) \to \mathbb{F}$ by $T(A) = \operatorname{tr}(A)$ (the trace of a 2×2 matrix). Compute the matrix of the transformation T in the standard bases of $M^{2\times 2}(\mathbb{F})$ and $\mathbb{F}$.

$$tr\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} + a_{22}.$$

(ordered) Basis of $M^{2\times 2}(\mathbb{F})$: $\beta = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

Basis of $\mathbb{F}$: $\gamma = (1)$.

$$[T]_\beta^\gamma = (1, 0, 0, 1).$$