Thu, Sept 4, 2014 **MATH 462, Quiz 1** Prof. V. Panferov

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

"$\Rightarrow$" By contradiction: assume

$W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

Then $\exists\ x_1 \in W_1,\ x_1 \notin W_2$ and

$\exists\ x_2 \in W_2,\ x_2 \notin W_1$, and so, $x_1, x_2 \in W_1 \cup W_2$.

Consider $x = x_1 + x_2$.

We claim that $x \notin W_1 \cup W_2$.

Indeed, if $x \in W_1$ then

$\underset{W_1}{\underset{\cap}{x}} + \underset{W_1}{\underset{\cap}{(-x_1)}} = x_2 \Rightarrow x_2 \in W_1$ – contradiction.

if $x \in W_2$ then $\underset{W_2}{\underset{\cap}{x}} + \underset{W_2}{\underset{\cap}{(-x_2)}} = x_1 \Rightarrow x_1 \in W_2$ – contradiction.

Thus $x \notin W_1 \cup W_2$, $W_1 \cup W_2$ is not closed for addition.

"$\Leftarrow$" If $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ – a subspace of V.

If $W_2 \subseteq W_1$ then $W_1 \cup W_2 = W_1$ – a subspace of V.

In either case $W_1 \cup W_2$ is a subspace of V.

2. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

No. One way:

$$(c_1 + c_2)(a_1, a_2) = ((c_1+c_2)a_1, a_2)$$

$$c_1(a_1, a_2) + c_2(a_1, a_2) = ((c_1+c_2)a_1, a_2^2)$$

Take $a_2 \neq 0, 1$ then $a_2 \neq a_2^2$, so one of the distributive properties fails.

Another way: $0 \cdot (a_1, a_2) = (0, a_2)$, which produces different answers depending on a_2, contradicting the uniqueness of zero.

3. Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ generate $\mathbb{R}^3$.

Show that the system of equations

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

has solutions for every a, b, c.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & c \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 2 & c+b-a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(b-c+a) \\ 0 & 1 & 0 & \frac{1}{2}(a-b+c) \\ 0 & 0 & 1 & \frac{1}{2}(c+b-a) \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} b-c+a \\ a-b+c \\ c+b-a \end{pmatrix} \quad \checkmark$$