Name: (print) ____

Each problem is worth 2 points. Show all your work.

1. In this problem $\mathbb{Q}$ denotes the set of all rational numbers.

   (a) Prove that the set of real numbers $\mathbb{R}$ (with the usual operations) is a vector space over $\mathbb{Q}$.

   (b) Prove that the sets

   $$\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Q}\} \quad \text{and} \quad \mathbb{Q}[\sqrt{2}, \sqrt{3}] = \{a+b\sqrt{2}+c\sqrt{3} \in \mathbb{R} : a, b, c \in \mathbb{Q}\}$$

   are subspaces of $\mathbb{R}$. Find their bases and dimensions.

   (c) Prove that the vector space $\mathbb{R}$ over $\mathbb{Q}$ is infinite-dimensional.

Continued...

2. Consider the set P consisting of all *positive* real numbers. For $x,\ y \in P$ and $\lambda \in \mathbb{R}$ we introduce operations

$$x \oplus y = xy, \qquad \lambda \odot x = x^{\lambda}.$$

Is the set P with operations $\oplus$ and $\odot$ a vector space? If yes, find a basis and determine the dimension of P.

Continued...

3. (a) Consider the vector space $M^{2\times 2}(\mathbb{R})$ consisting of all 2×2 matrices with real entries. Find a basis $\{A_1, A_2, A_3, A_4\}$ for $M^{2\times 2}(\mathbb{R})$ such that $A_i^2 = A_i$ for each i. *[Remark: Square matrices A satisfying $A^2 = A$ are known as projection matrices.]*

   (b) Find an analogous basis for the vector space $M^{3\times 3}(\mathbb{R})$.

Continued...

4. Let W be a subspace of $\mathbb{C}^3$ spanned by $u_1 = (1, 0, i)$ and $u_2 = (1 + i, 1, -1)$.

   (a) Show that u_1 and u_2 form a basis for W.

   (b) Show that the vectors $v_1 = (1, 1, 0)$ and $v_2 = (1, i, 1+i)$ are in W and form another basis for W.

   (c) What are the coordinates of u_1 and u_2 in the ordered basis v_1, v_2?

The end.