November 19, 2019 MATH 450A Prof. V. Panferov

**Midterm 2 Review Questions**

1. Find the supremum and infimum of the set $S$. Give a proof:

   (a) $S = \{x : x^2 - 4x + 3 < 0\}$

   (b) $S = \{s_n : s_n = \sum_{i=1}^{n} \frac{(-1)^i}{3^i},\ n \in \mathbb{N}\}$.

2. If $A, B$ are nonempty subsets of $\mathbb{R}$ such that $\forall\, x \in A\ \forall\, y \in B\ x \le y$, prove that $\sup A = \inf B$ if and only if $\forall\, \varepsilon > 0\ \exists\, x_\varepsilon \in A\ \exists\, y_\varepsilon \in B$ such that $y_\varepsilon - x_\varepsilon < \varepsilon$.

3. Prove that $f(x) = x \sin \frac{\pi}{x}$ is uniformly continuous on $(0, 1]$. Is $f$ uniformly continuous on $(0, \infty)$?

4. Prove that $f : x \mapsto x^{1.01}$ is not uniformly continuous on $[1, \infty)$.

5. If $x_n \in (a, b)$ is Cauchy, $f : (a, b) \to \mathbb{R}$ is uniformly continuous, prove that $f(x_n)$ is Cauchy.

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f$ is not differentiable at any $x_0 \in \mathbb{R}$, however $f^2$ is differentiable at every $x_0 \in \mathbb{R}$.

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f$ and $f^2$ are not differentiable at $x = 0$, however $f^3$ is differentiable at every $x_0 \in \mathbb{R}$.

8. Find all real $\alpha$ such that

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   is differentiable at 0.

9. Let

$$f(x) = \begin{cases} x(1-x) + e^{-\frac{1}{x^2}}, & x > 0 \\ x, & x \le 0. \end{cases}$$

   Find $f'(0)$ and $f''(0)$ if they exist.

10. (see also problems 5.1: 18, 19) Suppose $f$ is continuous at $x_0$ and such that $f(x_0) \neq 0$. Show there exist constants $r > 0$ and $c_0 > 0$ such that $|f(x)| \ge c_0$ for $|x - x_0| < r$.

11. Find the lower and the upper Darboux sums for the function $f(x) = 1 - 2x$ over the interval $[0, 1]$ using the partition $P_n$ with $n$ equal subintervals. Compute the limits $n \to \infty$.

12. (see Theorem 5.6) Prove that if $f$ is continuous on $[a, b]$ then for any partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ there exist points $x_i \in I_i$ such that

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i)\,\Delta x_i,$$

where $\Delta x_i = t_i - t_{i-1}$, $I_i = [t_{i-1}, t_i]$.

13. (see problem 5.1.17) Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Prove that $f$ is integrable on $[-1, 1]$ and that $F(x) = \int_{-1}^{1} f(t)\,dt$ is differentiable on $(-1, 1)$. Find $F'(0)$.

14.* (Fermat) Compute the integral $\int_1^2 x^p\,dx$ as a limit of Riemann sums using points in $[1, 2]$ that form a geometric progression (left or right end points of the intervals could be used).

15.* (see also problem 5.1.15) The function

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q, \text{ an irreducible fraction} \\ 0, & \text{otherwise} \end{cases}$$

is called the Riemann function on $\mathbb{R}$. Show that $f$ is integrable on any interval $[a, b]$ and that $\int_a^b f(x)\,dx = 0$.