Midterm 2 Review Questions

1. Find the supremum and infimum of the set \( S \). Give a proof:
   (a) \( S = \{ x : x^2 - 4x + 3 < 0 \} \)
   (b) \( S = \{ s_n : s_n = \sum_{i=1}^{n} \frac{(-1)^i}{3^i}, n \in \mathbb{N} \} \).

2. If \( A, B \) are nonempty subsets of \( \mathbb{R} \) such that \( \forall x \in A \forall y \in B x \leq y \), prove that \( \sup A = \inf B \) if and only if \( \forall \varepsilon > 0 \exists x_\varepsilon \in A \exists y_\varepsilon \in B \) such that \( y_\varepsilon - x_\varepsilon < \varepsilon \).

3. Prove that \( f(x) = x \sin \frac{\pi}{x} \) is uniformly continuous on \( (0, 1] \). Is \( f \) uniformly continuous on \( (0, \infty) \)?

4. Prove that \( f : x \mapsto x^{1.01} \) is not uniformly continuous on \( [1, \infty) \).

5. If \( x_n \in (a, b) \) is Cauchy, \( f : (a, b) \to \mathbb{R} \) is uniformly continuous, prove that \( f(x_n) \) is Cauchy.

6. Give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) is not differentiable at any \( x_0 \in \mathbb{R} \), however \( f^2 \) is differentiable at every \( x_0 \in \mathbb{R} \).

7. Give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) and \( f^2 \) are not differentiable at \( x = 0 \), however \( f^3 \) is differentiable at every \( x_0 \in \mathbb{R} \).

8. Find all real \( \alpha \) such that
   \[ f(x) = \begin{cases} 
   x^\alpha \cos \frac{\pi}{x}, & x \neq 0 \\
   0, & x = 0 
   \end{cases} \]
   is differentiable at \( 0 \).

9. Let
   \[ f(x) = \begin{cases} 
   x(1 - x) + e^{-\frac{1}{x}}, & x > 0 \\
   x, & x \leq 0. 
   \end{cases} \]
   Find \( f'(0) \) and \( f''(0) \) if they exist.

10. (see also problems 5.1: 18, 19) Suppose \( f \) is continuous at \( x_0 \) and such that \( f(x_0) \neq 0 \).
    Show there exist constants \( r > 0 \) and \( c_0 > 0 \) such that \( |f(x)| \geq c_0 \) for \( |x - x_0| < r \).

11. Find the lower and the upper Darboux sums for the function \( f(x) = 1 - 2x \) over the interval \([0, 1]\) using the partition \( P_n \) with \( n \) equal subintervals. Compute the limits \( n \to \infty \).
12. (see Theorem 5.6) Prove that if \( f \) is continuous on \([a, b]\) then for any partition \( P = \{t_0, t_1, \ldots, t_n\} \) of \([a, b]\) there exist points \( x_i \in I_i \) such that

\[
\int_a^b f(x) \, dx = \sum_{i=1}^n f(x_i) \Delta x_i,
\]

where \( \Delta x_i = t_i - t_{i-1}, \ I_i = [t_{i-1}, t_i] \).

13. (see problem 5.1.17) Let

\[
f(x) = \begin{cases} 
  x \sin \frac{1}{x}, & x \neq 0 \\
  1, & x = 0 
\end{cases}
\]

Prove that \( f \) is integrable on \([-1, 1]\) and that \( F(x) = \int_{-1}^x f(t) \, dt \) is differentiable on \((-1, 1)\). Find \( F'(0) \).

14. (Fermat) Compute the integral \( \int_1^2 x^p \, dx \) as a limit of Riemann sums using points in \([1, 2]\) that form a geometric progression (left or right end points of the intervals could be used).

15. (see also problem 5.1.15) The function

\[
f(x) = \begin{cases} 
  1/q, & \text{if } x = p/q, \ \text{an irreducible fraction} \\
  0, & \text{otherwise}
\end{cases}
\]

is called the Riemann function on \( \mathbb{R} \). Show that \( f \) is integrable on any interval \([a, b]\) and that \( \int_a^b f(x) \, dx = 0 \).