Tue, Dec 3, 2019 MATH 450A Quiz 9

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. For what values p does the series $\sum_{n=2}^{\infty} \frac{(\log n)^p}{n}$ converge? Determine all such values.

(According to textbook and mathematical tradition, $\log n = \ln n$ (natural log.))

Let $f(x) = \frac{(\log x)^p}{x}$; then $f'(x) = \frac{p(\log x)^{p-1}\frac{1}{x} \cdot x - (\log x)^p}{x^2}$

$$= (\log x)^{p-1} \frac{p - \log x}{x^2} < 0$$

if $\log x > p$, i.e. $x > e^p$

$\Rightarrow$ $f(x)$ is decreasing for $x > e^p$.

$f(x)$ is continuous. (combination of elementary functions.)

Then $\sum_{n=2}^{\infty} \frac{(\log n)^p}{n}$ conv $\Longleftrightarrow$ $\int_2^{\infty} \frac{(\log x)^p}{x}\,dx$ conv.

(since the sum $\sum_{n=2}^{[e^p]} \frac{(\log n)^p}{n}$ is finite.)

$$\int_2^{\infty} \frac{(\log x)^p}{x}\,dx = \int_{\log(2)}^{\infty} u^p\,du = \left[\frac{u^{p+1}}{p+1}\right]_{\log(2)}^{\infty}$$

$\begin{bmatrix} u = \log x \\ du = \frac{1}{x}dx \end{bmatrix}$ $= \lim_{u \to \infty} \frac{u^{p+1}}{p+1} - \frac{(\log(2))^{p+1}}{p+1}$

$(p \neq -1)$

$\lim_{u \to \infty} \frac{u^{p+1}}{p+1}$ is finite $\Longleftrightarrow$ $p < -1$

If $p = -1$ then $\int_2^{\infty} \frac{(\log x)^{-1}}{x}\,dx = \Big[\log(\log(x))\Big]_2^{\infty} = \infty$

— diverges

Therefore $\sum_{n=2}^{\infty} \frac{(\log n)^p}{n}$ conv $\Longleftrightarrow$ $p < -1$.

Please turn over...

2. Find all the values x for which the given power series converges:

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1}. \quad = \sum_{n=0}^{\infty} u_n(x)$$

Ratio test: $\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \left|\frac{3^{n+1} x^{n+1} (n+1)}{(n+2)\, 3^n x^n}\right| = \left(\frac{n+1}{n+2}\right) 3|x| \xrightarrow[n\to\infty]{} 3|x|$

Since $\frac{n+1}{n+2} = \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \to 1$

Thus if $3|x|<1$ $(|x|<\frac{1}{3})$ then the series conv. abs.

Let $|x|=\frac{1}{3}$. If $x=\frac{1}{3}$ then we have $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges since $\int_1^{\infty} \frac{1}{x}\,dx = \infty$

If $x=-\frac{1}{3}$ then we have $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges conditionally (alternating series test.)

Thus the series conv. for $x \in [-\frac{1}{3}, \frac{1}{3})$.

3. Test the series for convergence or divergence. If convergent, determine whether the series converges absolutely or conditionally:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

Ratio Test: $\left|\frac{(-1)^n (n+1)! \quad 1\cdot 3\cdot 5 \cdots (2n-1)}{1\cdot 3\cdot 5 \cdots (2n+1)(-1)^{n-1}\, n!}\right| =$

$$= \left|\frac{-(n+1)}{(2n+1)}\right| = \frac{n+1}{2n+1} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$$

Since $\frac{n+1}{2n+1} = \frac{1+\frac{1}{n}}{2+\frac{1}{n}} \to \frac{1}{2}$

By the Ratio Test the series converges absolutely.