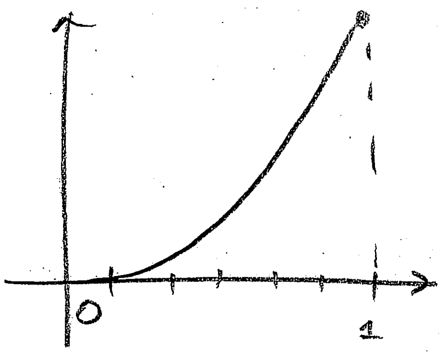
Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Compute $S^+(f, P_n)$ and $S^-(f, P_n)$ for the function $f(x) = x^2$ defined on $I = [0, 1]$, where P_n is the partition of I into n intervals of equal size. Show that

$$S^+(f, P_n) - S^-(f, P_n) \to 0, \quad \text{as } n \to \infty.$$



$$S^-(f, P_n) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3} \sum_{i=1}^{n-1} i^2$$

$$S^+(f, P_n) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$m_i = f(t_{i-1})$

$M_i = f(t_i)$

$t_i = \frac{i}{n}$; $i = 0 \ldots n$

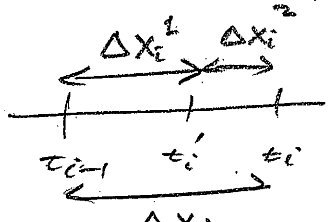
$\Delta x_i = \frac{1}{n}$

$$S^+(f, P_n) - S^-(f, P_n) = \frac{n^2}{n^3} = \frac{1}{n} \to 0$$

as $n \to \infty$.

2. If f is strictly increasing on $I = [a,b]$ show that $S^+(f, P') < S^+(f, P)$ where P' is any refinement of P. [Hint: Consider the case when P' is obtained from P by adding a single point first.]

If $P = \{t_0 \dots t_n\}$, $P' = \{t_0, \dots t_{i-1}, t_i', t_i, \dots t_n\}$



Then $$S^+(f,P') - S^+(f,P) = \Big[\sup_{[t_{i-1},\, t_i']} f\Big](t_i' - t_{i-1}) + \Big[\sup_{[t_i',\, t_i]} f\Big](t_i - t_i') - \Big[\sup_{[t_{i-1},\, t_i]} f\Big](t_i - t_{i-1})$$

$$= f(t_i')\,\Delta x_i^1 + f(t_i)\,\Delta x_i^2 - f(t_i)\,\Delta x_i$$

$$= \underbrace{\big(f(t_i') - f(t_i)\big)}_{<0}\underbrace{\Delta x_i^1}_{>0} < 0$$

Generally, consider $P = P_0' \to P_1' \to \dots \to P_k' = P'$ where each P_i' is obtained from P_{i-1}' by adding 1 point.

Then $S^+(f,P') - S^+(f,P) = \sum_{i=1}^{k} \underbrace{S^+(f,P_i') - S^+(f,P_{i-1}')}_{<0} < 0.$

3. Suppose that f is a bounded function on $I = [a,b]$. Let $M = \sup f$ and $m = \inf f$ for $x \in I$. Also, define $M^* = \sup|f|$ and $m^* = \inf|f|$ for $x \in I$. Show that

$$M^* - m^* \le M - m.$$

__Case 1__: If $f \ge 0$ on $[a,b]$ then $f = |f|$

$\Rightarrow M^* - m^* = M - m$

__Case 2__: If $f \le 0$ on $[a,b]$ then $f = -|f|$

$\Rightarrow M^* - m^* = -\inf f + \sup f =$

$= M - m.$

__Case 3__. If $M = \sup f > 0$ and $m = \inf f < 0$

Then

$M^* - m^* \le M^* = M \le M - m$ since $m < 0$