Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Suppose that f is continuous, increasing function on a closed interval $[a, b]$. Show that the range of f is the interval $[f(a), f(b)]$.

f increasing $\Rightarrow$ $\forall x > a \quad f(x) > f(a)$

$\forall x < b \quad f(x) < f(b)$

$\Rightarrow \forall x \in (a,b) \quad f(a) < f(x) < f(b)$

$\Rightarrow \forall x \in [a,b] \quad f(a) \leq f(x) \leq f(b).$

$\Rightarrow f([a,b]) \subseteq [f(a), f(b)].$

Now suppose $y \in [f(a), f(b)]$ then

If $y = f(a)$ or $y = f(b)$

$y \in f([a,b])$.

Otherwise $f(a) < y < f(b)$, so

by the Intermediate Value Theorem

$\exists\, c \in (a,b) \quad f(c) = y.$

Therefore $y \in f((a,b))$

$\Rightarrow y \in f([a,b])$.

This shows that $[f(a), f(b)] \subseteq f([a,b])$.

2. Determine whether the sequence x_n is a Cauchy sequence. If it is not, find at least one Cauchy subsequence x_{n_k}. Verify the Cauchy property for this subsequence by definition:

$$x_n = (1 + (-1)^n)n + \frac{1}{n}.$$

x_n is not a Cauchy sequence, since it contains a divergent subsequence

$$x_{2k} = 2\cdot(2k) + \frac{1}{2k} = 4k + \frac{1}{2k} \geq 4k \to \infty.$$

However

$$x_{2k-1} = 0\cdot(2k-1) + \frac{1}{2k-1} = \frac{1}{2k-1} \to 0$$

Cauchy property: Given $\varepsilon > 0$, if $l > k$

$$|x_{2k-1} - x_{2l-1}| = \left|\frac{1}{2k-1} - \frac{1}{2l-1}\right| = \left|\frac{(2l-1)-(2k-1)}{(2k-1)(2l-1)}\right|$$

$$= \frac{|(2l-1)-(2k-1)|}{|2l-1|}\,\frac{1}{2k-1} \leq \frac{1}{2k-1} < \varepsilon$$

if $2k-1 > \frac{1}{\varepsilon}$. Take $K > \frac{1}{\varepsilon}$, then $l > k > K$ $\Rightarrow |x_{2k-1} - x_{2l-1}| < \varepsilon$.

3. Show that the function $f : x \mapsto 1/x$ is uniformly continuous on $[1, \infty)$.

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \frac{|x_1 - x_2|}{|x_1||x_2|}$$

$$\leq |x_1 - x_2| < \delta = \varepsilon$$

since $|x_1|, |x_2| \geq 1$.

Given $\varepsilon > 0$ let $\delta = \varepsilon$ then

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$