Tue, Oct 15, 2019 **MATH 450A** **Quiz 5**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Let $f(x) = a_m x^m + \ldots + a_1 x + a_0$ be a polynomial of odd degree. (i) Use the Intermediate Value Theorem to show that $f(x) = 0$ has at least one real root. (ii) Show that the range of f is $\mathbb{R}$.

(i) We assume $m = 2k-1$, $a_m \neq 0$.

Consider $g(x) = \frac{f(x)}{a_m} = x^m + \ldots + b_1 x + b_0$

$= x^m \left(1 + \frac{b_{m-1}}{x} + \ldots + \frac{b_0}{x^m}\right)$

$= x^m h(x).$

Since $\lim_{x \to \pm\infty} h(x) = 1$, $\lim_{x \to \pm\infty} x^m = \pm\infty$

$\Rightarrow \lim_{x \to +\infty} g(x) = +\infty$, $\lim_{x \to -\infty} g(x) = -\infty$

$\Rightarrow \exists x_2 > 0$, $g(x_2) > 0$, $\exists x_1 < 0$ $g(x_1) < 0$

Since g is continuous on $[x_1, x_2]$

by IVT $\exists c \in (x_1, x_2)$, $g(c) = 0$

$\Rightarrow f(c) = a_m g(c) = 0.$

(ii) Given $y \in \mathbb{R}$ consider $F(x) = f(x) - y$

Then $F(x)$ is a polynomial of odd degree, so by part (i)

$\exists c \in \mathbb{R}$, $F(c) = 0$

$\Rightarrow f(c) - y = 0$

$\Rightarrow f(c) = y$

Thus $\forall y \in \mathbb{R}$ $\exists c \in \mathbb{R}$ $f(c) = y \Rightarrow f(\mathbb{R}) = \mathbb{R}$.

Please turn over...

2. Given that $S = \{x \in \mathbb{R} : x^2 - 2x - 3 < 0\}$, show that $\inf(S) = -1$.

$x^2 - 2x - 3 = (x+1)(x-3)$

$x - 2x - 3 < 0 \iff x+1 > 0,\ x - 3 < 0$

$\iff x > -1,\ x < 3.$

Thus, -1 is a lower bound for S

Suppose -1 is not an infimum of S

Then $\exists\, \varepsilon > 0$ such that $-1+\varepsilon$ is a lower bound for S.

However the number $x_\varepsilon = \min\{0, -1+\frac{\varepsilon}{2}\}$ satisfies $x_\varepsilon > -1$, $x_\varepsilon \leq 0 < 3 \Rightarrow x_\varepsilon \in S$,

$x_\varepsilon \leq -1 + \frac{\varepsilon}{2} < -1 + \varepsilon$

Thus, $-1+\varepsilon$ is not a lower bound for S $\Rightarrow \inf(S) = -1$.

3. Given that

$$S = \{s_n : s_n = 1 + \sum_{j=1}^{n} \frac{1}{j!},\ n \in \mathbb{N}\}$$

show that the set S has 3 as its upper bound.

$S = \{2,\ 2+\frac{1}{2},\ 2+\frac{1}{2}+\frac{1}{6},\ \dots\}$

$= \{2,\ 2\frac{1}{2},\ 2\frac{2}{3},\ \dots\}$

Claim: $s_n \leq 3 - \frac{1}{n}$, $n \in \mathbb{N}$.

By the previous line, this is true for $n = 1, 2, 3$.

Suppose for $k \in \mathbb{N}$ $s_k \leq 3 - \frac{1}{k}$

Then $s_{k+1} = s_k + \frac{1}{(k+1)!} \leq s_k + \frac{1}{(k+1)k}$

$\leq 3 - \frac{1}{k} + \frac{1}{(k+1)k} = 3 - \frac{k+1-1}{(k+1)k}$

$= 3 - \frac{1}{k+1}.$

By induction, $s_n \leq 3 - \frac{1}{n}$, $n \in \mathbb{N}$.

$\Rightarrow s_n \leq 3$.