Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Determine whether the function is (i) continuous from the left (ii) continuous from the right (iii) continuous at $x = a$:

$$f(x) = \begin{cases} (x^2-1)/(x^4-1), & 1 < x < 2 \\ x^2+3x-2, & 2 \le x < 5, \end{cases} \qquad a = 2.$$

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{x^2-1}{(x^2-1)(x^2+1)}$$

$$= \lim_{x\to 2-} \frac{1}{x^2+1} = \frac{1}{\lim_{x\to 2-} x \cdot \lim_{x\to 2-} x + 1} = \frac{1}{5}$$

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} x^2+3x-2$$

$$= \lim_{x\to 2+} x \cdot \lim_{x\to 2+} x + \lim_{x\to 2+} 3 \lim_{x\to 2+} x - \lim_{x\to 2+} 2$$

$$= 4 + 6 - 2 = 8$$

$$f(2) = x^2+3x-2\Big|_{x=2} = 8$$

$f(x)$ is continuous from the right

$\left(f(2) = \lim_{x\to 2+} f(x)\right)$

$f(x)$ is not continuous from the left

$\Rightarrow$ not continuous at $x=2$.

Please turn over...

2. Prove that

$$\lim_{x\to\infty} (x^2+1)/x^{3/2} = +\infty.$$

$$f(x) = \frac{x^2+1}{x^{3/2}} = x^{\frac{1}{2}} + x^{-\frac{3}{2}} \geq x^{\frac{1}{2}}$$

Given $A > 0$

$$f(x) \geq x^{\frac{1}{2}} > A \quad \text{if } x > A^2$$

Therefore, take $B = A^2$

$\forall A > 0 \quad \exists B = A^2 \quad \forall x > B$

$$f(x) > A.$$

3. Give an example of functions $f(x)$ and $g(x)$ such that $f(x) \to +\infty$, $g(x) \to -\infty$ as $x \to \infty$, and

$$\lim_{x\to\infty} f(x) + g(x) = A,$$

A an arbitrary real number.

Examples: (i) $f(x) = |x|$; $g(x) = A - |x|$

Then $f(x) \to +\infty$, $g(x) \to -\infty$

but $f(x) + g(x) = A$

(ii) $f(x) = \sqrt{x^2 + \alpha|x|}$; $g(x) = -|x|$

$$\lim_{x\to\infty} \sqrt{x^2+\alpha|x|} - |x|$$

$$= \lim_{x\to\infty} \frac{\alpha|x|}{\sqrt{x^2+\alpha|x|} + |x|} = \frac{\alpha}{2} = A$$

if $\alpha = 2A$