Thu, Sep 19, 2019 **MATH 450A** **Quiz 3**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Determine a value $\delta > 0$ so that for the given value of ε, the statement "$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$" is valid:

$$f(x) = (x^3 + 1)/(x + 1), \quad a = -1, \quad L = 3, \quad \varepsilon = 0.01.$$

$$\left|\frac{x^3+1}{x+1} - 3\right| = |x^2 - x + 1 - 3| = |x^2 - x - 2| = |x+1|\,|x-2|$$

$$< \delta |x-2| \leq \delta \cdot 4 \leq \varepsilon$$

if $\delta \leq 1$ so $|x+1| \leq 1 \Rightarrow |x-2| \leq 4$

Take $\delta = \min\left\{1, \frac{\varepsilon}{4}\right\} = 0.0025$ when $\varepsilon = 0.01$

then both conditions are satisfied and $|f(x) - 3| < \varepsilon$.

2. Prove by induction: if $a > 0$ and $n \in \mathbb{N}$ then $(1 + a)^n \geq 1 + na$.

Given $a > 0$, let $S = \{n \in \mathbb{N} : (1+a)^n \geq 1 + na\}$

Since $(1+a)^1 = 1 + 1 \cdot a$, then $1 \in S$.

Suppose $k \in S$, then $(1+a)^k \geq 1 + ka$.

$$\Rightarrow (1+a)^{k+1} = (1+a)^k (1+a) \geq (1+ka)(1+a)$$

$$= 1 + ka + a + ka^2$$

$$\geq 1 + ka + a = 1 + (k+1)a$$

$\Rightarrow k + 1 \in S$.

Therefore S is inductive $\Rightarrow S = \mathbb{N}$.

3. Suppose f and h are continuous at a and $f(x) \leq g(x) \leq h(x)$ for $|x - a| < k$. If $f(a) = h(a)$ show by an ε-δ argument that g is continuous at a.

Given $\varepsilon > 0$ $\quad \exists\, \delta_1 > 0,\ \delta_2 > 0$:

$|f(x) - f(a)| < \varepsilon$ when $|x - a| < \delta_1$

and $|h(x) - h(a)| < \varepsilon$ when $|x - a| < \delta_2$.

Take $\delta = \min\{\delta_1, \delta_2\}$. Then

$|x - a| < \delta \implies$

$$\left.\begin{aligned} f(a) - \varepsilon < f(x) < f(a) + \varepsilon \\ h(a) - \varepsilon < h(x) < h(a) + \varepsilon \end{aligned}\right\} \quad \text{and } f(a) = h(a)$$

$$\implies f(a) - \varepsilon < f(x) \leq g(x) \leq h(x) < f(a) + \varepsilon$$

$$\implies |g(x) - f(a)| < \varepsilon$$

Since also $f(a) \leq g(a) \leq h(a) = f(a)$

$$\implies g(a) = f(a)$$

$$\implies |g(x) - g(a)| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $g(x)$ is continuous at $x = a$.