1 (a)

$$0\cdot a + a \underset{(M4)}{=} 0\cdot a + 1\cdot a \underset{(D)}{=} (0+1)\cdot a \underset{(A4)}{=} 1\cdot a \underset{(M4)}{=} a$$

$$\Rightarrow \quad (0\cdot a + a) + (-a) = a + (-a)$$

$\binom{A1}{A5}$

$$\underset{(A3)}{\Rightarrow} \quad 0\cdot a + (a + (-a)) = a + (-a)$$

$$\underset{(A5)}{\Rightarrow} \quad 0\cdot a + 0 = 0 \underset{(A4)}{\Rightarrow} 0\cdot a = 0$$

(b) $$(-1)a + a \underset{(M4)}{=} (-1)a + 1\cdot a$$
$$\underset{(D)}{=} ((-1)+1)\cdot a \underset{\binom{A2}{A5}}{=} 0\cdot a \underset{\binom{\text{part}}{(a)}}{=} 0$$

Therefore $(-1)a + a = (-a) + a$

as in part (a)
$$((-1)a + a) + (-a) = (-a + a) + (-a)$$
$$(-1)a + (a + (-a)) = (-a) + (a + (-a))$$
$$(-1)a + 0 = -a + 0$$
$$(-1)a = -a.$$

(c) $$(-a)c + ac \underset{(D)}{=} ((-a) + a)c \underset{(A5)}{=} 0\cdot c \underset{(\text{part } a)}{=} 0$$

Therefore $(-a)c = -ac$

Also $a + (-a) = -(-a) + (-a) = 0$
$$\Rightarrow -(-a) = a$$

$$\Rightarrow (-a)(-b) = -(a(-b)) = -((-b)a)$$
$$= -(-ba) = ba = ab.$$

(d) By contradiction: If $1<0$

then $-1>0$

$$\Rightarrow (-1)(-1) = 1\cdot 1 = 1 > 0$$

(set of positive # is closed under mult.)

$\Rightarrow$ contradiction: can't be both $1<0$ and $1>0$.

#2. (a) Given $a>0$.

$a^1 = 1+(a-1)$ (true equality).

Let $S = \{n \in \mathbb{N} : a^n \geq 1+n(a-1)\}$

Then $1 \in S$;

If $n \in S$ then
$$a^{n+1} = a^n \cdot a$$
$$\geq (1+n(a-1))a$$
$$= a + na^2 - na$$
$$\geq a + na - n - 1 + 1$$
$$= (n+1)(a-1)+1$$

Therefore S is inductive

$\Rightarrow S = \mathbb{N}$.

(b) If $a > 1$ then $a^n \geq 1 + n(a-1)$

$$\geq n(a-1) \longrightarrow \infty$$

Therefore $a^n \longrightarrow \infty$

$\Big(\forall A > 0 \ \exists N \in \mathbb{N}$ $\left(N > \frac{A}{a-1} \text{ - positive \#} \right)$

$\forall n > N \quad a^n \geq n(a-1) \geq N(a-1) > A \Big)$

If $0 < b < 1$ then set $a = \frac{1}{b} \Rightarrow a > 1$

Then $\left(\frac{1}{b}\right)^n \longrightarrow \infty$

$\frac{1}{b^n} \longrightarrow \infty \Rightarrow b^n \longrightarrow 0$

$\Big(\forall \varepsilon > 0$ let N be such that

$\frac{1}{b^n} > \frac{1}{\varepsilon}, \ n > N$

$\Rightarrow b^n < \varepsilon, \ n > N \Big)$

#3(a) Let $n \in \mathbb{N}$.

First of all $\forall x_1, x_2$

$$x_1 < x_2 \iff x_1^n < x_2^n$$

(By induction, if $n=1$, trivially true;
if $x_1^n < x_2^n$ then $x_1^{n+1} = x_1 \cdot x_1^n < x_1 \cdot x_2^n$
$< x_2 \cdot x_2^n = x_2^{n+1}$
and similarly with the opposite inequality.

Let $S = \{x^n : 0 < x < 1\}$
then $\forall y \in S$ $y = x^n \Rightarrow 0 < y < 1$
Thus 1 is an upper bound for S,
and 0 is a lower bound for S.
Suppose for $\varepsilon > 0$, $1-\varepsilon$ is an upper bd.
Then $x^n < 1-\varepsilon$ for all x: $0<x<1$.
$\iff x < (1-\varepsilon)^{\frac{1}{n}}$
where $\alpha = (1-\varepsilon)^{\frac{1}{n}}$ is the number such that $\alpha^n = 1-\varepsilon$.
Since $1-\varepsilon < 1$ we must have $\alpha < 1$
Take α' such that $\alpha < \alpha' < 1$
Then $0 < \alpha'^n < 1$, so $\alpha'^n \in S$
but $\alpha'^n > \alpha^n = 1-\varepsilon$
$\Rightarrow 1-\varepsilon$ is not an upper bound for S $\Rightarrow 1 = \sup(S)$ (least upper bound.)
(contradiction)

Similarly, let $\varepsilon > 0$ be a lower bound for S.

Then $\forall x : 0 < x < 1 \Rightarrow x^n \geq \varepsilon$

$\Rightarrow \forall x : 0 < x < 1 \Rightarrow x \geq \varepsilon^{\frac{1}{n}}$

but $\varepsilon^{\frac{1}{n}} > 0$ because $\varepsilon > 0$

so take $x : 0 < x < \varepsilon^{\frac{1}{n}} \Rightarrow x^n < \varepsilon$

$\Rightarrow \varepsilon$ is not a lower bound;

contradiction.

Therefore $0 = \inf(S)$ (the greatest lower bound.

(b) $$S_n = \sum_{j=1}^{n} \frac{1}{2^j} = \sum_{j=1}^{n} \left(\frac{1}{2}\right)^j = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n \longrightarrow 1$$

as $n \to \infty$

Since $\left(\frac{1}{2}\right)^n \xrightarrow[n \to \infty]{} 0$.

Also $S_n \leq 1$ for $n = 1, 2, \ldots$

Therefore $\sup(S) = 1$:

an upper bound such that

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N$

$-\varepsilon < S_n - 1 < 0$

$\Rightarrow 1 - \varepsilon$ is not an upper bound for $\varepsilon > 0$

$\Rightarrow 1 = \sup(S)$.

Since $s_n = 1-\left(\frac{1}{2}\right)^n$ is an increasing sequence,

$\inf(S) = \min(S) = s_1 = \frac{1}{2}$.

($\forall n \quad 1-\left(\frac{1}{2}\right)^n \geq 1-\left(\frac{1}{2}\right)^1 = \frac{1}{2}$

and $s_1 = \frac{1}{2}$

therefore $\frac{1}{2} + \underset{(\varepsilon>0)}{\varepsilon}$ is not a lower bound for S

$\Rightarrow \frac{1}{2} = \inf(S)$)

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#4: "$\Rightarrow$" Suppose $A, B \neq \emptyset$, $\forall x \forall y$ $x \in A, y \in B$ $\Rightarrow x \leq y$;

$\alpha = \sup(A) = \inf(B)$.

Let $\varepsilon > 0$ be given.

Since $\alpha - \frac{\varepsilon}{2}$ is not an upper bound for B,

$\exists x_\varepsilon \in A$ $x_\varepsilon > \alpha - \frac{\varepsilon}{2}$.

Since $\alpha + \frac{\varepsilon}{2}$ is not a lower bound for B,

$\exists y_\varepsilon \in B$: $y_\varepsilon < \alpha + \frac{\varepsilon}{2}$

Therefore $\alpha - \frac{\varepsilon}{2} < x_\varepsilon \leq y_\varepsilon < \alpha + \frac{\varepsilon}{2}$

$\Rightarrow y_\varepsilon - x_\varepsilon < \left(\alpha + \frac{\varepsilon}{2}\right) - \left(\alpha - \frac{\varepsilon}{2}\right) = \varepsilon$

"$\Leftarrow$" Suppose A, B : $\forall x \forall y$ $x \in A$ $y \in B$ $\Rightarrow x \leq y$

$\forall \varepsilon > 0$ $\exists x_\varepsilon \in A$ $\exists y_\varepsilon \in B$ $y_\varepsilon - x_\varepsilon < \varepsilon$.

Then $A \neq \emptyset$, bounded above (by any element of B)

$B \neq \emptyset$, bounded below (by any element of A).

$\Rightarrow \sup(A), \inf(B)$ are finite,

any $y \in B$ is an upper bd of $A \Rightarrow$ $\forall y \in B$ $\sup(A) \leq y \Rightarrow$ $\sup(A) \leq \inf(B)$

and $\forall \varepsilon > 0$ $x_\varepsilon \leq \sup(A) \leq \inf(B) \leq y_\varepsilon$

$\Rightarrow \sup(A) - \inf(B) < y_\varepsilon - x_\varepsilon < \varepsilon$

$\Rightarrow \sup(A) = \inf(B)$ since $\varepsilon > 0$ is arbitrary.

#6. (a) $f(x) = \sqrt{x}$; $I = [0, \infty)$

WTS: $|f(x_2) - f(x_1)| = |\sqrt{x_2} - \sqrt{x_1}| < \varepsilon$ if $|x_2 - x_1| < \delta$

(By contr.) Suppose $x_2 > x_1$, $\sqrt{x_2} - \sqrt{x_1} \geq \varepsilon$

$$\Rightarrow \sqrt{x_2} \geq \sqrt{x_1} + \varepsilon$$

$$\Rightarrow x_2 \geq (\sqrt{x_1} + \varepsilon)^2$$

$$\Rightarrow x_2 \geq x_1 + 2\varepsilon\sqrt{x_1} + \varepsilon^2$$

$$\Rightarrow x_2 - x_1 \geq 2\varepsilon\sqrt{x_1} + \varepsilon^2 \geq \varepsilon^2$$

So, if $x_2 - x_1 < \varepsilon^2$ then $\sqrt{x_2} - \sqrt{x_1} < \varepsilon$

Take $\delta = \varepsilon^2$ then $|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \delta$

$\Rightarrow$ unif. cont.

(b) $f(x) = x^{1.01}$; $I = (1, \infty)$

WTS: $\exists \varepsilon_0 > 0 \quad \forall \delta > 0 \quad \exists x_1, x_2 \in (1, \infty)$ $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| \geq \varepsilon_0$.

Let $x_1 = n$, $x_2 = n + \alpha$, $\alpha < \delta$.

$$|f(x_2) - f(x_1)| = (n+\alpha)^{1.01} - n^{1.01} = n^{1.01}\left(\left(1 + \frac{\alpha}{n}\right)^{1.01} - 1\right)$$

but $\left(1 + \frac{\alpha}{n}\right)^{1.01} \geq 1 + 1.01\frac{\alpha}{n}$

$$\Rightarrow |f(x_2) - f(x_1)| \geq n^{1.01} \cdot 1.01\frac{\alpha}{n} = 1.01\,\alpha\, n^{0.01}$$

Let $\alpha = \frac{1}{n^{0.01}}$, then as $n \to \infty$ $\alpha \to 0$ so we can make $\alpha <$ any prescribed δ, but

$$1.01\,\alpha\, n^{0.01} = 1.01 > 1.$$

Negation of unif. cont. works with $\varepsilon_0 = 1$, $x_1 = n$, $x_2 = n + \alpha$.

#10.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\alpha}\cos\frac{\pi}{h}}{h}$$

$$= \lim_{h\to 0} h^{\alpha-1}\cos\frac{\pi}{h}$$

If $\alpha > 1$ then $\lim_{h\to 0} h^{\alpha-1} = 0$

and since

$$-h^{\alpha-1} \le h^{\alpha-1}\cos\frac{\pi}{h} \le h^{\alpha-1}$$

$$\lim_{h\to 0} h^{\alpha-1}\cos\frac{\pi}{h} = 0$$

If $\alpha = 1$ $\lim_{h\to 0} \cos\frac{\pi}{h}$ does not exist

since $h_n = \frac{1}{2n} \Rightarrow \cos\frac{\pi}{h_n} = 1$

$h_n = \frac{1}{2n+1} \Rightarrow \cos\frac{\pi}{h_n} = -1$

If $\alpha < 1$ then $\lim_{h\to 0} h^{\alpha-1} = \pm\infty$

and the example of the above sequences shows that $\lim_{h\to 0} h^{\alpha-1}\cos\frac{\pi}{h}$ does not exist.

Answer: differentiable $\iff \alpha > 1$.

\# 11. (a)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Then f is discontinuous at every $x_0 \in \mathbb{R}$

(because if $x_0 \in \mathbb{Q}$, $f(x_0) = 1$
but $\forall \delta > 0$ the interval $(x_0 - \delta, x_0 + \delta)$
contains irrational values x for
which $|f(x) - f(x_0)| = 2$

and if $x_0 \in \mathbb{R}\setminus\mathbb{Q}$ $f(x_0) = -1$,
but $\forall \delta > 0$ the interval $(x_0 - \delta, x_0 + \delta)$
contains rational values x for
which $|f(x) - f(x_0)| = 2$)

Therefore f is not differentiable
at any $x_0 \in \mathbb{R}$

but $f^2(x) = 1$, $x \in \mathbb{R} \Rightarrow$ differentiable.

(b) $f^3(x) = (f(x))^3 = x$

$f(x) = x^{\frac{1}{3}}$; $f^2(x) = (f(x))^2 = x^{\frac{2}{3}}$

Then $f^3(x)$ is diff. everywhere
(the identity function.)

$$f'(0) = \lim_{h \to 0} \frac{h^{1/3} - 0^{1/3}}{h} = \lim_{h \to 0} h^{-2/3} = \infty$$

(so derivative DNE $\Rightarrow f$ not diff at 0)

$$(f^2)'(0) = \lim_{h \to 0} \frac{h^{2/3} - 0^{2/3}}{h} = \lim_{h \to 0} h^{-1/3} = \pm\infty$$

(so derivative DNE $\Rightarrow f^2$ not diff. at 0)

#13. Let $P = \{t_0, t_1, \ldots t_n\}$

where $a = t_0 < t_1 < \ldots < t_n = b$.

Then by MVT for integrals

$\forall i = 1 \ldots n \quad \exists c_i \in I_i = [t_{i-1}, t_i]$

$$f(c_i)(t_i - t_{i-1}) = \int_{t_{i-1}}^{t_i} f(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(x)\,dx$$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

$(\Delta x_i = t_i - t_{i-1})$

#14. Suppose, by contradiction, that $\exists x_0 \in (a,b), f(x_0) \neq 0$

Suppose $f(x_0) > 0$, otherwise consider $-f$ instead of f.

Since f is continuous $\exists\, \delta > 0$ such that

$[x_0 - \delta, x_0 + \delta] \subseteq [a, b)$, and $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$

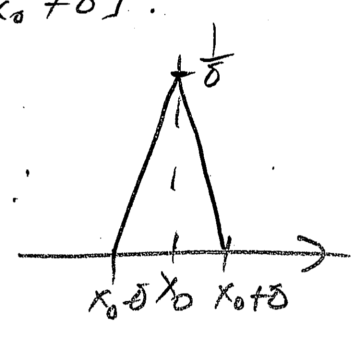
$$\underset{f(x_0) - f(x)}{\rotatebox{90}{$\leq$}} \Rightarrow f(x) \geq f(x_0) - \frac{f(x_0)}{2}$$

$$= \frac{f(x_0)}{2} =: \alpha > 0$$

Also since f is continuous,

$f(x) \geq \alpha$ on $[x_0 - \delta, x_0 + \delta]$.

Let $g(x) = \begin{cases} \frac{1}{\delta^2}(x - (x_0 - \delta)), & x_0 - \delta < x < x_0 \\ -\frac{1}{\delta^2}(x - (x_0 + \delta)), & x_0 < x < x_0 + \delta \\ 0, & \text{otherwise} \end{cases}$



Rmk: Not true if f is only assumed integrable on $[a,b]$: Example: $f(x) = \begin{cases} 1, & x = \frac{a+b}{2} \\ 0, & \text{otherwise} \end{cases}$

Then
$$\int_a^b f(x)g(x)\,dx \geq \alpha \int_a^b g(x)\,dx = \alpha \cdot 1 = \alpha > 0$$

which contradicts the statement

$$\int_a^b f(x)g(x)\,dx = 0.$$

therefore $f(x) = 0$ on (a,b)

$\Rightarrow$ by continuity $f(x) = 0$ on $[a,b]$.

#15.
$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= -\left(1 + \cot^2 x\right).$$

arccot x is the inverse function

for $\cot : (0, \pi) \to \mathbb{R}$

so $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$, and

$$(\operatorname{arccot} y)' = \frac{1}{\cot'(x)} = -\frac{1}{1+\cot^2(x)}$$
$$= -\frac{1}{1+y^2}.$$

by the Theorem about the derivative of the inverse function

(Note that $1 + \cot^2 x \neq 0$ for $x \in \mathbb{R}$)

#16. $\forall x \in [0,1)$ $\quad x^n \xrightarrow[n\to\infty]{} 0$

$$\left(x^n \le \frac{1}{1+\left(\frac{1}{x}-1\right)n} = \frac{1}{n}\,\frac{1}{\frac{1}{n}+\left(\frac{1}{x}-1\right)} \xrightarrow[n\to\infty]{} 0 \right)$$

If $x=1$, $x^n = 1$, so $x^n \xrightarrow[n\to\infty]{} 1$

Thus, pointwise on $[0,1]$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} =: f(x).$$

However

$$|f_n(x) - f(x)| = \begin{cases} x^n, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

and $\displaystyle\sup_{[0,1]} |f_n(x) - f(x)| = 1 \underset{n\to\infty}{\not\to} 0$

(see Problem 2)

Therefore the convergence is not uniform

#17. (a) $$\sum_{n=1}^{\infty} \frac{(3/2)^n x^n}{n+1}$$

Ratio test:

$$\left| \frac{(3/2)^{n+1} x^{n+1}}{n+2} \Big/ \frac{(3/2)^n x^n}{n+1} \right| = \frac{3}{2}|x| \frac{n+1}{n+2} \xrightarrow[n\to\infty]{} \frac{3}{2}|x| < 1$$

when $|x| < \frac{2}{3}$

$\Rightarrow$ the series conv. abs. for $|x| < \frac{2}{3}$.

If $x = \frac{2}{3}$ the series is $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges (harmonic series.)

If $x = -\frac{2}{3}$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges (alternating series.)

Answer: $x \in \left[-\frac{2}{3}, \frac{2}{3}\right)$.

(b) $$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log^2 n} \left(\frac{x}{2}\right)^n$$

Ratio test: $$\left| \frac{(-1)^{n+1} \left(\frac{x}{2}\right)^{n+1}}{(n+1)\log^2(n+1)} \frac{n \log^2 n}{(-1)^n \left(\frac{x}{2}\right)^n} \right|$$

$$= \frac{|x|}{2} \frac{n}{n+1} \left(\frac{\log n}{\log(n+1)}\right)^2 \longrightarrow \frac{|x|}{2}$$

since $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$

$$\log n - \log(n+1) = \log \frac{n}{n+1} \xrightarrow[n\to\infty]{} \log 1 = 0$$

$$\Rightarrow \log(n+1) = (\log n) + \alpha_n \quad \text{where } \alpha_n \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \frac{\log n}{\log(n+1)} = \frac{\log n}{(\log n) + \alpha_n}$$

$$= \frac{1}{\log n + \frac{\alpha_n}{\log n}} \xrightarrow[n\to\infty]{} 1$$

$$\frac{\alpha_n}{\log n} \xrightarrow[n\to\infty]{} 0$$

Thus, the series conv. abs for

$$\frac{|x|}{2} < 1 \iff |x| < 2$$

If $x = 2$ the series is $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log^2 n}$

converges – alternating

If $x = -2$ the series is $\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$

Since $\int_2^{\infty} \frac{1}{x \lg^2 x}\, dx = \int_{\log 2}^{\infty} \frac{1}{u^2}\, du$

$$= \left[-\frac{1}{u}\right]_{\log 2}^{\infty} = \frac{1}{\log 2} \text{ conv}$$

and $\left(\frac{1}{x \lg^2 x}\right)' = \frac{-(\log^2 x + 2 \log x)}{(x \log^2 x)^2} < 0$ for $x > 1$

by integral test

$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$ conv.

Answer: conv. for $x \in [-2, 2]$.

#18. $f(x) = \ln(1+x) \longrightarrow f(0) = 0$

$f' = \frac{1}{1+x}$

$f'' = -\frac{1}{(1+x)^2}$

$f''' = \frac{1\cdot 2}{(1+x)^3}$

$f^{(4)} = -\frac{1\cdot 2\cdot 3}{(1+x)^4}$

$f^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$

$(n \geq 1)$

$$p_n(x;0) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{(j-1)!}{j!} x^j$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{x^j}{j}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \dots + (-1)^{n-1} \frac{x^n}{n}.$$

$$|R_n(x;0)| = \left| \frac{1}{n!} \int_0^x (x-t)^n (-1)^n \frac{n!}{(1+t)^{n+1}} dt \right|$$

$$= \left| \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} dt \right|$$

$$= |x|^{n+1} \int_0^1 \frac{(1-s)^n}{(1+xs)^{n+1}} ds$$

Since $1-s \leq 1+xs$, $\quad x \in [-1,1]$

then $0 \leq \frac{1-s}{1+xs} \leq 1 \implies \quad s \in [0,1]$

$$|R_n(x;0)| \leq |x|^{n+1} \int_0^1 \frac{1}{1+xs} ds$$

$$= |x|^{n+1} \frac{\ln(1+x)}{x} = |x|^n |\ln(1+x)|.$$

Let $0 < h < 1$.

$\forall x \in [-h, h]$

$$\left| R_n(x,0) \right| \le |x|^n \left| \ln(1+x) \right| \le h^n \left| \ln(1+h) \right| \xrightarrow[n \to \infty]{} 0$$

Since $h^n \to 0$ for $0 < h < 1$.

Therefore $\sup\limits_{[-h,h]} \left| R_n(x,0) \right|$

$$= \sup_{[-h,h]} \left| p_n(x) - f(x) \right| \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow \quad p_n \rightrightarrows f$ on $[-h, h]$.

#19 (b)

$$f = \cos x \qquad f(0) = 1$$
$$f' = -\sin x \qquad f'(0) = 0$$
$$f'' = -\cos x \qquad f''(0) = -1$$
$$f''' = \sin x \qquad f'''(0) = 0$$
$$f^{(4)} = \cos x \qquad f^{(4)}(0) = 1$$
$$\vdots \qquad \vdots$$

$$p_{2n}(x;0) = \sum_{j=0}^{2n} \frac{f^{(j)}(0)}{j!} x^j \qquad \left(\begin{array}{l} f^{(j)}(0) \neq 0 \\ \text{for } j \\ \text{even only} \end{array} \right)$$

$$= \sum_{j=0}^{n} \frac{f^{(2j)}(0)}{(2j)!} x^{2j}$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} x^{2j}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + \frac{(-1)^n}{(2n)!} x^{2n}$$

$$p_{2n+1}(x;0) = p_{2n}(x,0) \quad \text{since } f^{(2n+1)}(0) = 0$$

$$|p_{2n}(x;0) - f(x)| = |R_{2n}(x;0)|$$

$$\underset{(\text{Lagrange})}{=} \left| \frac{f^{(2n+1)}(\xi)}{(2n+1)!} x^{2n+1} \right| \leq \frac{1}{(2n+1)!} |x|^{2n+1}$$

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)!} |x|^{2n+1} \quad \text{converges for every } x \in \mathbb{R}$$

Since by ratio test,

$$\left| \frac{1}{(2n+3)!} |x|^{2n+3} \frac{(2n+1)!}{|x|^{2n+1}} \right| = \frac{|x|^2}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0$$

Therefore $\forall x \in \mathbb{R}$

$$|p_{2n}(x;0) - f(x)| \xrightarrow[n\to\infty]{} 0 .$$

(b) If $x \in [-M, M] \Rightarrow |x| \leq M$
$\Rightarrow |x|^{2n+1} \leq M^{2n+1}$

$$\Rightarrow |p_{2n}(x;0) - f(x)| = \frac{1}{(2n+1)!} M^{2n+1} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \sup |p_{2n}(x;0) - f(x)| \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ $p_{2n}(x;0)$ converge to $f(x)$ uniformly on any interval of the form $[-M, M]$.