MATH 450A Final Exam Review Problems

December 14, 2019

1. Prove using Axioms of Real Numbers:

   (a) $0a = 0$.

   (b) $(-1)a = -a$,

   (c) $(-a)(-b) = ab$.

   (d) $1 > 0$

   Indicate which of the axioms is used in each step.

2. (a) Prove that if $a > 1$ and $n \in \mathbb{N}$ then $a^n \geq 1 + n(a-1)$.

   (b) Prove that if $a > 1$ then $a^n \to \infty$, and if $0 < b < 1$ then $b^n \to 0$.

3. (a) Given $n \in \mathbb{N}$, prove that $\sup\{x^n : 0 < x < 1\} = 1$ and $\inf\{x^n : 0 < x < 1\} = 0$.

   (b) Find the supremum of the set $S = \{s_n : s_n = \sum_{j=1}^{n} (1/2^j), n \in \mathbb{N}\}$. Give a proof.

4. If A, B are nonempty subsets of $\mathbb{R}$ such that $\forall x \in A \ \forall y \in B \ x \leq y$, prove that $\sup A = \inf B$ if and only if $\forall \varepsilon > 0 \ \exists x_\varepsilon \in A \ \exists y_\varepsilon \in B$ such that $y_\varepsilon - x_\varepsilon < \varepsilon$.

5. Given $\varepsilon > 0$ find δ in the definition of limit: $\lim_{x \to a} f(x) = L$,

$$f(x) = \frac{x}{x+1}, \quad a = 3, \quad L = 3/4.$$

6. (a) Prove that the function $f : x \mapsto \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

   (b) Prove that $f : x \mapsto x^{1.01}$ is not uniformly continuous on $[1, \infty)$.

7. Determine whether or not the function is continuous at the given value a. If it is not continuous decide whether or not the function is continuous on the left or on the right.

   (a) $f(x) = x(1 + 1/x^2)^{1/2}$, $x \neq 0$, $f(0) = 1$, $a = 0$.

   (b) $f(x) = \frac{1}{1+e^{\frac{1}{x-1}}}$, $x \neq 1$, $f(1) = 1$, $a = 1$.

8. Consider the sequence

$$x_n = \sum_{j=1}^{n} \frac{1}{j!}.$$

   Give *five* different proofs to show that x_n is convergent.

9. If $x_n \in (a, b)$ is Cauchy, $f : (a, b) \to \mathbb{R}$ is uniformly continuous, prove that $f(x_n)$ is Cauchy.

10. Find all real α such that

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   is differentiable at 0.

11. (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at any $x_0 \in \mathbb{R}$, however f^2 is differentiable at every $x_0 \in \mathbb{R}$.

    (b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f and f^2 are not differentiable at $x = 0$, however f^3 is differentiable at every $x_0 \in \mathbb{R}$.

12. Suppose that f is integrable on $I = [a, b]$ and that $0 < m \leq f(x) \leq M$ for $x \in I$. Prove that $\int_a^b (1/f(x))\, dx$ exists.

13. Suppose that f is continuous on $I = [a, b]$. Prove that for any partition P of $[a, b]$ there exists a Riemann sum corresponding to this partition such that the value of the sum is *equal* the value of the integral $\int_a^b f(x)\, dx$.

14. Suppose that f is continuous on $I = [a, b]$ and that $\int_a^b f(x)g(x)\, dx = 0$ for every function g continuous on I. Prove that $f(x) = 0$ on I. Does the statement remain true if f is only assumed to be integrable?

15. Using that $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$ show that

$$\operatorname{arccot}'(x) = \frac{-1}{1+x^2}.$$

16. Show that the sequence $f_n : x \mapsto x^n$ converges for each $x \in [0, 1]$ but the convergence is not uniform.

17. Determine all x for which the series converge:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(3/2)^n x^n}{n+1} \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{(-1)^n}{n \log^2 n} \left(\frac{x}{2}\right)^n \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{n!\, x^n}{n^{n+1}}.$$

18. For the function $f(x) = \ln(1 + x)$ find the Taylor polynomial $p_n(x) = p_n(x; x_0)$ of order n about $x_0 = 0$. If

$$f(x) = p_n(x) + R_n(x),$$

    show that $|R_n(x)| \leq |x|^n |f(x)|$ for $-1 < x < 1$. Deduce that the Taylor series for the function f at 0 converges to f uniformly on every interval $[-h, h]$ with $0 < h < 1$.

19. In each of the examples below find the Taylor series about $a = 0$ and prove that it converges <u>to $f(x)$</u> (a) pointwise on $(-\infty, \infty)$; (b) uniformly on $[-M, M]$, with $M > 0$.

$$\text{(a)}\ f(x) = \sin x, \qquad \text{(b)}\ f(x) = \cos x, \qquad \text{(c)}\ f(x) = e^x,$$

20. Suppose f and f' are continuous in an interval I containing 0. Prove that for any a in I

$$|f(0)| \leq \frac{1}{a} \int_0^a |f(x)|\, dx + \int_0^a |f'(x)|\, dx.$$