Thu, Nov 21, 2019 **MATH 450A, Midterm 2** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test includes 8 questions, on 8 pages. The perfect score is 42 points; the last question is a bonus worth an extra 6 points. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. No electronic devices are permitted. Show all your work.

1. (6 points) Find the supremum and the infimum of the set

$$S = \{y : y = x/(x+1),\ x \geq 0\}.$$

Give a proof.

$x \geq 0 \Rightarrow x+1 > 0 \Rightarrow \frac{x}{x+1} \geq 0 \Rightarrow 0$ is a lower bound for S.

$0 < 1 \Rightarrow x < x+1$; since $x+1 > 0 \Rightarrow \frac{x}{x+1} < 1 \Rightarrow 1$ is an upper bound for S.

$0 \in S$ since $\frac{0}{0+1} = 0 \Rightarrow 0 = \min(S) \Rightarrow 0 = \inf(S)$

Let $f : x \mapsto \frac{x}{x+1}$. Then $S = f([0,\infty))$

If $y \in [0,1)$ then $\frac{x}{x+1} = y \Leftrightarrow$

$x = y(x+1) \Leftrightarrow$

$(1-y)x = y \Leftrightarrow$

$x = \frac{y}{1-y}$

Therefore $S \supseteq [0,1)$, and since $S \subseteq [0,1]$ and $1 \notin S$ we must have $S = [0,1)$.

Since $\forall \varepsilon > 0$ $x_\varepsilon = 1 - \min\{1, \frac{\varepsilon}{2}\} > 1 - \varepsilon$ and $x_\varepsilon \in [0,1)$ then $1-\varepsilon$ is not an upper bound for $S \Rightarrow 1 = \sup(S)$.

2. (6 points) If x_n is a nondecreasing sequence (*i. e.* $\forall n\ x_{n+1} \geq x_n$) and $\lim_{n\to\infty} x_n = a$ show that

$$\sup(x_n) = a.$$

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N \quad |x_n - a| < \varepsilon$

If for some $n = n_1$, $x_{n_1} > a$

then $x_{n_1+1} \geq x_{n_1} \Rightarrow x_{n_1+1} > a$

By induction, $x_{n_1+k} \geq x_{n_1} > a$

Thus for $n \geq n_1$ $|x_n - a| > |x_{n_1} - a|$,

a contradiction.

Therefore $x_n \leq a$ for all n.

Now if $\varepsilon > 0$ then $\exists N \in \mathbb{N}$

s.t. $-\varepsilon < x_{N+1} - a \leq 0$

$\Rightarrow x_{N+1} > a - \varepsilon$

$\Rightarrow a - \varepsilon$ is not an upper bound

for (x_n)

$\Rightarrow \sup(x_n) = a$.

3. (6 points) Prove or disprove:

$$f : x \mapsto x^3 - x \text{ is uniformly continuous on } [0, \infty).$$

f is not uniformly continuous on $[0,\infty)$

(Hint: any unif. cont. function on $[0,\infty)$ must satisfy $|f(x)| \le ax+b$)

WTS: $\exists\, \varepsilon_0 > 0 \quad \forall \delta > 0 \quad \exists\, x_1, x_2 \in [0,\infty)$

$|x_2 - x_1| < \delta$, $|f(x_2) - f(x_1)| \ge \varepsilon_0$.

Given $\delta > 0$ take x_1, x_2 large, but $|x_2 - x_1| < \delta$.

Let $x_1 = n$, $x_2 = n + \alpha$, $0 < \alpha < \delta$.

$|f(x_2) - f(x_1)| = |(n+\alpha)^3 - (n+\alpha) - n^3 + n|$

$= |\cancel{n^3} + 3n^2\alpha + 3n\alpha^2 + \alpha^3 - \cancel{n} - \alpha - \cancel{n^3} + \cancel{n}|$

$= \alpha \cdot |3n^2 + 3n\alpha + \alpha^2 - 1|$

Let n be such that $3n\alpha > 1$

then $3n^2 + 3n\alpha + \alpha^2 - 1 > 3n^2 > 3 \cdot \left(\frac{1}{3\alpha}\right)^2$

$\Rightarrow |f(x_2) - f(x_1)| > \alpha \cdot \frac{1}{3\alpha^2} = \frac{1}{3\alpha}$

Let $\alpha = \min\left\{\frac{1}{3}, \delta\right\}$, then $\frac{1}{3\alpha} \ge 1$

Therefore for $\varepsilon_0 = 1 \quad \forall \delta > 0$

$\exists\, x_1 = n, \; x_2 = n + \alpha$ such that

$|x_2 - x_1| < \delta$, $|f(x_2) - f(x_1)| \ge \varepsilon_0$

f is not uniformly continuous on $[0,\infty)$.

4. (6 points) Using the <u>definition</u> of a Cauchy sequence show that the following sequence is Cauchy:

$$x_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

<u>WTS</u>: $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \in \mathbb{N}$

$n > m > N \Rightarrow |x_n - x_m| < \varepsilon.$

Let $\varepsilon > 0$ be given, $n > m$.

$$|x_n - x_m| = \left| \sum_{k=m+1}^{n} \frac{(-1)^k}{k!} \right|$$

$$= \left| \frac{(-1)^{m+1}}{(m+1)!} + \underbrace{\frac{(-1)^{m+2}}{(m+2)!} + \dots + \frac{(-1)^n}{n!}}_{\leq 0} \right|$$

$$= \frac{1}{(m+1)!} \left| \underbrace{1 - \frac{1}{m+2} + \frac{1}{(m+2)(m+3)}}_{\leq 0} - \frac{1}{(m+2)(m+3)(m+4)} + \frac{1}{(m+2)(m+3)(m+4)(m+5)} + \dots \right.$$

$$\left. \dots + \frac{(-1)^{n-m-1}}{\underbrace{n!/(m+1)!}} \right|$$

≤ 0, or paired with the preceding negative term.

$$\leq \frac{1}{(m+1)!} \leq \frac{1}{m} < \varepsilon \quad \text{if} \quad m > \frac{1}{\varepsilon}$$

Let $N \in \mathbb{N}$ be such that $N \geq \frac{1}{\varepsilon}$

(exists by the Archimedian principle)

then $n > m > N \Rightarrow |x_n - x_m| < \varepsilon.$

5. (6 points) Show that the function

$$f(x) = \begin{cases} x\cos\frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

satisfies all conditions of Rolle's theorem on the interval $[0, 2]$ and that there are infinitely many values c in $(0, 2)$ such that $f'(c) = 0$.

[We assume that $\cos(x)$ is differentiable on $\mathbb{R}$ and $(\cos(x))' = -\sin(x)$.]

$f(0) = 0$ by defn. $\quad f(2) = 2\cdot\cos\left(\frac{\pi}{2}\right) = 0$

$$f'(x) = \cos\left(\frac{\pi}{x}\right) - x\left(\sin\left(\frac{\pi}{x}\right)\right)\cdot\left(-\frac{\pi}{x^2}\right)$$

$$= \cos\frac{\pi}{x} + \frac{\pi}{x}\sin\frac{\pi}{x}, \quad x \neq 0$$

$\Rightarrow f$ is diff. on $(0,\infty) \Rightarrow$ continuous on $(0,\infty)$

Also $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x\cos\frac{\pi}{x} = 0 = f(0)$

since $\left|x\cos\frac{\pi}{x}\right| \leq |x|$

$\Rightarrow f(x)$ is continuous from the right at $x=0$.

In fact, since $\cos\frac{\pi}{x} = 0$

when $\frac{\pi}{x} = \frac{\pi}{2} + \pi n \Rightarrow x = \frac{1}{\frac{1}{2}+n}$
$(n \in \mathbb{N})$

there are infinitely values in $(0,2)$ for which $f(x) = 0$.

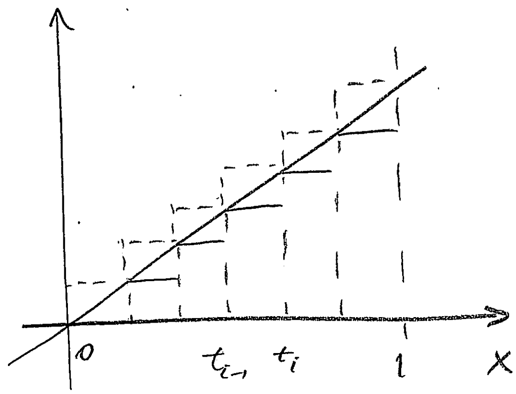
Applying Rolle's theorem to each interval $\left[\frac{1}{\frac{1}{2}+(n+1)}, \frac{1}{\frac{1}{2}+n}\right]$ we obtain infinitely many values c_n such that $f'(c_n) = 0$.

Continued...

6. (6 points) Let P_n denote the partition of the interval $[0, 1]$ into n equal sub-intervals and let $f(x) = x$. Compute the upper and the lower Darboux sums $S^+(f, P_n)$ and $S^-(f, P_n)$ and find their limit as $n \to \infty$.

$$\Delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$t_i = \frac{i}{n}$$



$$S^-(f, P_n) = \sum_{i=1}^{n} m_i \Delta x = \sum_{i=1}^{n} f(t_{i-1}) \Delta x = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n}(i-1) = \frac{1}{n^2}\sum_{i=1}^{n-1} i = \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{n-1}{2n}$$

$$S^+(f, P_n) = \sum_{i=1}^{n} M_i \Delta x = \sum_{i=1}^{n} f(t_i)\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{2n}.$$

$$\lim_{n\to\infty} S^+(f, P_n) = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} S^-(f, P_n) = \lim_{n\to\infty}\frac{n-1}{2n} = \lim_{n\to\infty}\frac{1-\frac{1}{n}}{2} = \frac{1}{2}$$

7. (6 points) (a) Let

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & \text{otherwise.} \end{cases}$$

(clearly, $\forall x \in \mathbb{R}$ $0 \le f(x) \le 1$)

Prove that f is not integrable on any interval $[a, b]$.

Let P be any partition of $[a, b]$.

$M_i = \sup_{[t_{i-1}, t_i]} f = 1$ since $\exists x \in \mathbb{Q}$, $t_{i-1} < x < t_i$ $\Rightarrow f(x) = 1$

$m_i = \inf_{[t_{i-1}, t_i]} f = 0$ since $\exists x \in \mathbb{R} \setminus \mathbb{Q}$, $t_{i-1} < x < t_i$ $\Rightarrow f(x) = 0$

$$S^-(f, P) = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$$

$$S^+(f, P) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a$$

$$\Rightarrow \sup_P S^-(f, P) = \underline{\int_a^b} f = 0 \neq 1 = \overline{\int_a^b} f = \inf_P S^+(f, P)$$

$\Rightarrow$ f is not integrable.

(b) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ such that f^2 is integrable, but f is not.

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & \text{otherwise} \end{cases}$$

Then $f^2(x) = 1$ — constant function $\Rightarrow$ integrable.

but $\overline{\int_0^1} f = 1$ as on part (a)

$\underline{\int_0^1} f = -1$

$\Rightarrow$ f is not integrable.

8. (bonus: 6 points) Suppose that f is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x(1-x)}}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the derivatives $f'(0)$, $f''(0)$ or prove they do not exist.

$f'_-(0) = 0$ since $f(x) = 0$ for $x < 0$.

$$f'_+(0) = \lim_{h\to 0+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0+} \frac{1}{h} e^{-\frac{1}{h}} e^{-\frac{1}{1-h}}$$

$$= \lim_{h\to 0+} \frac{1/h}{e^{1/h}} \underbrace{\lim_{h\to 0+} e^{-\frac{1}{1-h}}}_{=1} = \lim_{y\to\infty} \frac{y}{e^y}$$

$$= \lim_{y\to\infty} \frac{1}{e^y} = 0.$$

Thus $f'(0) = 0$

$$\Rightarrow \quad f'(x) = \begin{cases} \frac{1-2x}{x^2(1-x)^2} e^{-\frac{1}{x(1-x)}}, & 0 < x < 1 \\ 0, & x < 0 \end{cases}$$

$f''_-(0) = 0$ since $f'(x) = 0$ for $x < 0$

$$f''_+(0) = \lim_{h\to 0+} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0+} \frac{1-2h}{h^2(1-h)^2} e^{-\frac{1}{h}} e^{-\frac{1}{1-h}}$$

$$= \lim_{h\to 0+} \frac{1}{h^2} e^{-\frac{1}{h}} \underbrace{\lim_{h\to 0+} \frac{1-2h}{(1-h)^2}}_{=1} \underbrace{\lim_{h\to 0+} e^{-\frac{1}{1-h}}}_{=1}$$

$$= \lim_{y\to\infty} \frac{y^2}{e^y} = \lim_{y\to\infty} \frac{2y}{e^y} = \lim_{y\to\infty} \frac{2}{e^y} = 0$$

Therefore $f''(0) = 0$.