Thu, Oct 3, 2019 **MATH 450A, Midterm 1** Prof. V. Panferov

Name: (print) ______________________________

CSUN ID No. : Solutions.

This test includes 7 questions (40 points in total) in the main part, and one bonus question, worth an extra 6 points. Please check that your copy of the test has 8 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. No electronic devices are permitted during the exam. Show all your work.

1. (4 points) For any two sets A and B, prove that

$$(A \setminus B) \setminus C = A \setminus (B \cup C).$$

$\forall x$ $\quad x \in (A \setminus B) \setminus C$

$\iff x \in A \setminus B,\ x \notin C$ — Definition of "$\setminus$"

$\iff (x \in A,\ x \notin B),\ x \notin C$

$\iff x \in A,\ (x \notin B,\ x \notin C)$ — Associativity of "$\wedge$"

$\iff x \in A,\ \ x \notin B \cup C$ — De Morgan's law

$\iff x \in A \setminus (B \cup C)$ — Definition of "$\setminus$".

Thus, the two sets must have the same elements; by definition of set equality, they are equal.

2. (6 points) Prove, using Axioms of a Field. Give reasons for each step, referring to either one of the Axioms, or the previous part of the problem.

(a) If $a + x = b$ then $x = b + (-a)$.

$a + x = b$

comm. (A-2)

$\Rightarrow x + a = b$

well-defined addition (A-1)

$\Rightarrow (x + a) + (-a) = b + (-a)$

assoc. addition (A-3)

$\Rightarrow x + (a + (-a)) = b + (-a)$

Property of add. inverse (A-5)

$\Rightarrow x + 0 = b + (-a)$

Property of zero (A-4)

$\Rightarrow x = b + (-a)$

(b) $\underbrace{-(a + b)}_{x_1} = \underbrace{(-a) + (-b)}_{x_2}$.

$(a+b) + x_2 = (a+b) + ((-a) + (-b))$

comm (A-2)

$= (a+b) + ((-b) + (-a))$

assoc. add (A-3)

$= a + (b + ((-b) + (-a)))$

assoc. add (A-3)

$= a + ((b + (-b)) + (-a))$

Property of neg. (A-5)

$= a + (0 + (-a))$

Property of zero (A-4)

$= a + (-a)$

Property of neg (A-5)

$= 0$

By part (a)

$x_2 = 0 + (-(a+b))$

property of zero (A-4)

$= -(a+b) = x_1$

3. (6 points) Prove by induction:

(a) If $m \in \mathbb{N}$, $n \in \mathbb{N}$ then $m + n \in \mathbb{N}$.

Let $S = \{ n \in \mathbb{N} : \forall m \quad m \in \mathbb{N} \Rightarrow m+n \in \mathbb{N} \}$

Then $S \subseteq \mathbb{N}$,

$1 \in S$ since $\mathbb{N}$ is inductive ($\forall m \quad m \in \mathbb{N} \Rightarrow m+1 \in \mathbb{N}$)

If $k \in S$ then

$\forall m \quad m \in \mathbb{N} \Rightarrow m + (k+1) = \underbrace{(m+k)}_{\substack{\in \mathbb{N} \\ \text{since } k \in S}} + 1 \in \mathbb{N}$ since $\mathbb{N}$ is ind.

$\Rightarrow S$ is inductive $\Rightarrow S = \mathbb{N}$.

(b) If $0 < a < b$ and $n \in \mathbb{N}$ then $a^n < b^n$.

Given $a > 0$, $b > a$:

Let $S = \{ n \in \mathbb{N} : a^n < b^n \}$

Then $S \subseteq \mathbb{N}$,

$1 \in S$ since $a < b$

If $k \in S$ then $a^k < b^k$

then $a^{k+1} = a^k \cdot a < b^k \cdot a$
$< b^k \cdot b = b^{k+1}$

$\Rightarrow k+1 \in S$

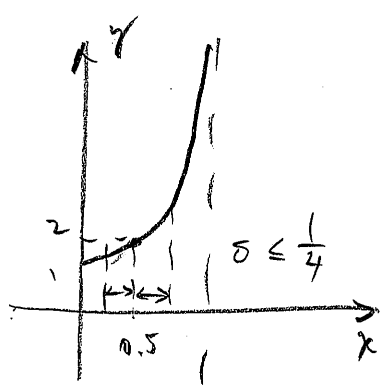
$\Rightarrow S$ is inductive $\Rightarrow S = \mathbb{N}$.

Continued...

4. (6 points) Given values of L, a and ε determine a value $\delta > 0$ such that the statement "$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$" is valid:

$$f(x) = \frac{x+1}{x-1}, \quad a = 0.5, \quad L = -3, \quad \varepsilon = 0.1.$$

$$|f(x) - L| = \left|\frac{x+1}{x-1} + 3\right| = \left|\frac{x+1+3(x-1)}{x-1}\right|$$



$$= \frac{|4x-2|}{|x-1|} = \frac{4\,|x-0.5|}{|x-1|} < \frac{4\delta}{|x-1|}$$

Suppose $\delta \le \frac{1}{4}$

Then $|x-1| \ge |x-0.5-0.5|$

$$\ge 0.5 - |x-0.5|$$

$$\ge 0.5 - \delta \ge 0.25$$

$$\Rightarrow \frac{1}{|x-1|} \le 4$$

$$\Rightarrow |f(x) - L| < \frac{4\delta}{|x-1|} \le 16 \cdot \delta \le \varepsilon$$

if $\delta = \min\left\{\frac{\varepsilon}{16}, \frac{1}{4}\right\}$.

$$= \frac{0.1}{16} = 0.00625$$

5. (6 points) Using an ε-δ-argument prove that the function $f : x \mapsto x^2$ is continuous at any $a \in \mathbb{R}$.

Given $\varepsilon > 0$,

$$|f(x) - f(a)| = |x^2 - a^2| = |(x-a)(x+a)|$$

$$= |x-a||x+a| < \delta |x+a|$$

$$= \delta |x - a + 2a|$$

$$\leq \delta (|x-a| + |2a|)$$

$$< \delta (\delta + |2a|)$$

Suppose $\delta \leq |a|$ then $\delta + |2a| \leq 3|a|$

then $|f(x) - f(a)| < 3|a|\delta \leq \varepsilon$

if $\delta = \min \left\{ \frac{\varepsilon}{3|a|}, |a| \right\}$.

6. (6 points) If $f(x) \to c$ and $g(x) \to \infty$ as $x \to +\infty$ ($c \in \mathbb{R}$) show that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0.$$

Use the "ε-δ"-technique.

$\forall \varepsilon > 0 \quad \exists A_1 > 0 \quad \forall x \quad x > A_1 \Rightarrow |f(x) - c| < \varepsilon$

$\forall B > 0 \quad \exists A_2 > 0 \quad \forall x \quad x > A_2 \Rightarrow |g(x)| > B$

Given $\varepsilon > 0$, $\left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} < \frac{|f(x)|}{B} \leq \frac{|c|+1}{B} = \varepsilon$

if $x > A_1$ such that $|f(x) - c| < 1$

$\Rightarrow |f(x)| = |f(x) - c + c| \leq |f(x) - c| + |c| < 1 + |c|$

and $x > A_2$ such that

$|g(x)| > B = \frac{|c|+1}{\varepsilon}$

Take $A = \max\{A_1, A_2\}$, then

$x > A \Rightarrow x > A_1,\ x > A_2 \Rightarrow$

$\left|\frac{f(x)}{g(x)}\right| < \varepsilon$

Thus $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0.$

7. (6 points)

(a) Prove that

$$\lim_{n\to\infty}\left(\frac{5}{n}-n\right)=-\infty.$$

Given $A>0$, $\frac{5}{n}-n < 5-n < -A$

if $n > A+5$

(b) Find the limit. Show all steps.

$$\lim_{n\to\infty}\frac{1}{n}(3n-\sin n).$$

$x_n = \frac{1}{n}(3n-\sin n) = 3 - \frac{\sin n}{n}$

$0 \le |x_n - 3| = \left|\frac{\sin n}{n}\right| \le \frac{1}{n} \longrightarrow 0$

(Archimedean principle.)

Therefore $x_n - 3 \longrightarrow 0$

$\Rightarrow x_n \longrightarrow 3.$

8. (bonus: 6 points) Find the limit $\lim_{n\to\infty} x_n$ (give a proof in each case):

(a) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$.

$$\underbrace{\frac{1}{\sqrt{n^2+n+1}} + \dots + \frac{1}{\sqrt{n^2+n+1}}}_{= n\cdot\frac{1}{\sqrt{n^2+n+1}}} \le x_n \le \underbrace{\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2}} + \dots + \frac{1}{\sqrt{n^2}}}_{= n\cdot\frac{1}{n} = 1}$$

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n+1}} = 1$,

by Sandwiching Principle $x_n \to 1$.

(b) $x_n = \frac{n}{2}\left(\sqrt[3]{1+\frac{2}{n}} - 1\right)$. $= \frac{n}{2} \frac{\left(\sqrt[3]{1+\frac{2}{n}} - 1\right)\left(\sqrt[3]{1+\frac{2}{n}} + \left(1+\frac{2}{n}\right)^{\frac{2}{3}} + 1\right)}{\left(1+\frac{2}{n}\right)^{\frac{2}{3}} + \left(1+\frac{2}{n}\right)^{\frac{1}{3}} + 1}$

$$= \frac{n}{2} \frac{\left(1+\frac{2}{n}\right) - 1}{\left(1+\frac{2}{n}\right)^{2/3} + \left(1+\frac{2}{n}\right)^{\frac{1}{3}} + 1}$$

$$= \frac{1}{\underset{\searrow 1}{\left(1+\frac{2}{n}\right)^{2/3}} + \underset{\searrow 1}{\left(1+\frac{2}{n}\right)^{1/3}} + 1} \xrightarrow[n\to\infty]{} \frac{1}{3}$$

since $\lim_{n\to\infty} \frac{2}{n} = 0$.

and $x \mapsto x^{2/3}$

and $x \mapsto x^{1/3}$ are continuous.

The end.