Problems on Uniform Continuity.

#1 True: if $f : [0,\infty) \to \mathbb{R}$ is continuous

and $\lim\limits_{x \to +\infty} f(x) = 0$ then f is unif. cont. on $[0,\infty)$.

Take an $\varepsilon > 0$.

Proof: Using that $\lim\limits_{x \to +\infty} f(x) = 0$

$\exists A > 1$ s.t. $|f(x)| < \frac{\varepsilon}{2}$

for $x > A-1$.

If $x_1, x_2 > A-1$ then

$|f(x_1) - f(x_2)| \le |f(x_1)| + |f(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Since f is unif. cont. on $[0, A]$

$\exists \delta_1 > 0 : \forall x_1, x_2 \in [0, A]$ $|x_1 - x_2| < \delta \Rightarrow$

$|f(x_1) - f(x_2)| < \varepsilon$

Now let $\delta = \min\{\delta_1, 1\}$.

If $|x_1 - x_2| < \delta$ then either

$x_1, x_2 \in (0, A]$ or

$x_1, x_2 \in (A-1, \infty)$. $\left(\begin{array}{c}\text{since } x_2 > A \Rightarrow \\ x_1 > x_2 - 1 > A-1\end{array}\right)$

In each case $|f(x_1) - f(x_2)| < \varepsilon$

$\Rightarrow f$ is unif. cont. on $[0, \infty)$.

#2. Suppose $f: [0, \infty) \to \mathbb{R}$ is unif. cont.

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Show that $|f(x)| \le ax + b$, $x \ge 0$ for some $a, b > 0$.

Soln: Let $\varepsilon_0 = 1 \Rightarrow \exists\, \delta_0 > 0$ s.t.

$$|x - x_1| < \delta_0 \Rightarrow |f(x_1) - f(x_2)| < 1$$

Thus, on $[0, \delta_0)$ $|f(x) - f(0)| < 1$

$$\Rightarrow |f(x)| < |f(0)| + 1$$

$\Rightarrow$ on $[0, \delta_0]$ $|f(x)| \le |f(0)| + 1$

(since $f(\delta_0) = \lim_{x \to \delta_0^-} f(x) \le |f(0)| + 1$)

Similarly, on $[\delta_0, 2\delta_0]$ $f(x) \le |f(\delta)| + 1 \le |f(0)| + 2$

and by induction,

$$|f(x)| \le |f(0)| + n, \quad \text{on } [(n-1)\delta_0, n\delta_0].$$

Now, for $x \in [(n-1)\delta_0, n\delta_0]$

$$n - 1 \le \frac{x}{\delta_0} \le n \Rightarrow n \le \frac{x}{\delta_0} + 1$$

Therefore $\forall n \in \mathbb{N}$

$$|f(x)| \le |f(0)| + 1 + \frac{x}{\delta_0}, \quad x \in [(n-1)\delta_0, n\delta_0]$$

$$\Rightarrow |f(x)| \le |f(0)| + 1 + \frac{x}{\delta_0}, \quad x \ge 0.$$

$$\left(b = |f(0)| + 1, \quad a = \frac{1}{\delta_0}\right)$$